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Dr. J. L. E. DREYER, PRESIDENT, in the Chair.

Harihara Subramani Aiyar, M.A., 118 Belsize Road, London, N.W. 6;
Rev. William Benjamin Hine, M.A., Chaplain and Science Master, Royal Savoy School, London, S.E. 22;
Herbert Horrocks, 1 Westminster Road, Chorley, Lancs;
Rev. C. B. Johnson, 134 Lorne Terrace, Preston New Road, Blackburn, Lancs;
Miss Katharine Pember, The Warden's Lodgings, All Souls' College, Oxford; and
Miss Elizabeth Williamson, Hill Hall, Theydon Mount, Epping, Essex,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Rev. Robert Armstrong, The Manse, Edgworth, Bolton, Lancs (proposed by W. R. Stokes);
George Elliott Barton, Gonville and Caius College, Cambridge (proposed by F. J. M. Stratton);
Arthur Frederick Bennett, M.Inst.C.E., Haylings, Leiston, Suffolk (proposed by W. H. Steavenson);
Reginald Robert Samuel Cox, Lecturer in Mathematics, University of Sheffield (proposed by R. H. Fowler);

Rev. John L. Gipprich, S.J., Director of the Observatory, Georgetown University, Washington, D.C., U.S.A. (proposed by A. L. Cortie);

Edward Harrison, M.D., M.A., F.R.C.S., St. Leonards, Hornsea, near Hull (proposed by A. L. Cortie);

Charles Hujer, R.N.C., University of Prague, and 26 Gloucester Road, London, N.W. 1 (proposed by H. Dingle);

C. A. Robert Lundin, Manager of Optical Dept., Alvan Clark & Sons, Cambridge, Mass., 173 School Street, Watertown, Mass., U.S.A. (proposed by J. H. Worthington);

George Pearson, Observatory House, Calton Hill, Edinburgh (proposed by J. M'D. Field);

Rev. Edward C. Phillips, S.J., Professor of Astronomy, Woodstock College, Woodstock, Md., U.S.A. (proposed by A. L. Cortie);

Percy Raymond Portman, Queen's College, Taunton, Somerset (proposed by H. Spencer Toy);

Archie Clyde Schock, A.M., Professor of Mathematics and Astronomy, Central High School, Philadelphia, Pa., U.S.A. (proposed by L. J. Comrie); and

Raymond Whitehead, B.Sc., Ivy Lea, Broad Road, Sale, Cheshire (proposed by H. Lowery).

Fifty-seven presents were announced as having been received since the last meeting, including, amongst others:—

F. Boquet: Histoire de l'Astronomie.

Carte photographique du ciel: Zone +12°, Bordeaux Observatory (6 charts); Zones +18°, +20°, +22°, Paris Observatory (12 charts).

Hyderabad, Nizamiah Observatory, Astrographic Catalogue, Vol. v. Dec. 21°.

R. Jarry-Desloges: Observations des Surfaces Planétaires, fasc. 6, années 1917, 1918.

G. R. Kaye: Hindu Astronomy.

La Plata Observatory: Publicaciones, Tome 8, Catalogo de 4412 estrellas.

Miss M. Proctor: Evenings with the Stars.

The Splendour of the Heavens (edited by Rev. T. E. R. Phillips and Dr. W. H. Steavenson).

The Equilibrium of the Calcium Chromosphere. By E. A. Milne.

1. *Scope of the Paper.*—In a paper recently communicated to the Society,* I have shown that the hypothesis of the equilibrium, under radiation pressure, of the ionised calcium in the high-level regions of the solar chromosphere, combined with Einstein's theory of radiation, leads to a determination of τ, the mean life of an excited atom of ionised calcium. The value of τ thus determined is of the order of magnitude found in the laboratory for other atoms, and is in agreement with theoretical estimates based on the theory of the structure of the atom.

The result was obtained by considering the conditions at the outer boundary of an atmosphere supported by radiation pressure. The investigation was, in fact, a fragment of the general investigation of the equilibrium of a Ca^+ atmosphere under radiation pressure. That investigation it is proposed to give here. The acceptableness of the value found for τ gives some confidence that the results obtained, in particular the somewhat remarkable law of density-distribution, afford a correct idea of the actual state of things in the high-level chromosphere. The investigation also throws light on the question of the residual intensities in absorption lines and on the relation of the reversing layer to the chromosphere.

The investigation has, however, a wider significance. The problem solved is that of the equilibrium under radiation pressure of an atmosphere consisting of atoms supposed to possess only two stationary states, and the solution is obtained in full. Now the *general* problem of the equilibrium of an atmosphere under radiation pressure is soluble in principle when the spectral properties of the atoms composing it are known; for we are then acquainted with each type of atomic event, namely, each type of absorption and emission, which can cause an atom to experience the pressure of radiation. But the solution would be excessively complicated. If there are n stationary states to take into account, there will be in general $(n-1)$ independent equations expressing that the gas is in a steady state, each containing $2(n-1)$ terms referring to the possible † transitions between a given state and the $(n-1)$ other states. Further, the equation of mechanical equilibrium will contain $\frac{1}{2}n(n-1)$ terms for the pressure of radiation. The labour involved in the solution of such a set of equations appears to be prohibitive, and I have been unable to make useful progress even with the boundary condition for atoms possessing three stationary states. With two stationary states the solution becomes manageable. We are thus compelled to rely on the case of two stationary states for indications as to the general characteristics of an atmosphere supported by radiation pressure. A little consideration shows that this special case will give a fair representation of any case in which the spectral lines involved are derived from the normal state of the atom. Such are the lines of

* *Monthly Notices*, 84, 354, 1924.

† The correspondence principle must, however, be taken into account, and this will reduce the number of possible transitions.

the principal series of Na, Ca, Ca^+, Sr^+, Ba^+. But for an atom like hydrogen, where the important lines are not derived from the normal state, the solution may be considerably different.

2. *Outline of Procedure and Results.*—Consider an atmosphere consisting of a gas whose atoms are capable of two, and only two, stationary states. These states are the "normal" state and the "excited" state. Transitions from one to the other are accompanied with the absorption or emission of quanta of amount $h\nu$. The absorption of a quantum $h\nu$ results in the acquisition of momentum $h\nu/c$, where c is the velocity of light; the emission of a quantum results in the acquisition of recoil momentnm $h\nu/c$. The resultant recoil momentum during a short interval will be zero, since the emissions will be distributed at random. Consequently, the pressure of radiation arises solely from the absorption of quanta $h\nu$.

Assuming that radiation having a continuous spectrum is incident on the atmosphere from below, the spectrum of the emergent radiation will be a continuous spectrum crossed by a single absorption line. Let us further idealise the problem by assuming that this absorption line is sharply bounded * and has a definite breadth $\Delta\nu$, and that inside the line the residual intensity is constant and equal to $(I_\nu)_0$. Then $(I_\nu)_0$ is the intensity of ν-radiation emerging from the atmosphere.

It is of importance to note the mechanism of formation of the absorption line. Of each of the beams of radiation incident respectively from below and above on the atoms in any given layer of the atmosphere, a certain fraction is absorbed. Of the amount absorbed, one half is re-radiated forwards and the other half is re-radiated backwards. The beam passing upwards, being the stronger, suffers a net depletion of energy in traversing the layer, since it provides more than one half of the total amount of energy absorbed but only receives half back. On the contrary, the beam passing downwards, being the weaker, suffers a net accession of energy. The function of each elementary layer of atoms is thus to deflect backwards some of the outward-moving radiation. A gradient is thus set up in the intensity of radiation, positive inwards, with the result that the emergent intensity is less than the incident, and an "absorption line" is produced. But it must be clearly understood that there is no ultimate absorption of energy. In the steady state, the energy by which the continuous spectrum is depleted in virtue of the formation of the absorption line is represented by the energy radiated back by the atmosphere to the material beneath it.

At any particular level in the atmosphere, let I_ν be the average intensity of the stream of ν-radiation crossing from below to above, I'_ν the average intensity of the stream from above to below. The net flux of energy from below to above is then $\pi(I_\nu - I'_\nu)\Delta\nu$; we write this $\pi F_\nu \Delta\nu$, so that

$$F_\nu = I_\nu - I'_\nu.$$

We now make the assumption that the atmosphere is in a steady state. Then there is no accumulation of energy at any point, and further, under the conditions of a stellar atmosphere, there can be no

* This assumption is removed in § 16 below.

generation of energy at any point. Hence the net flux of radiation must be the same from level to level, and so F_ν is a constant, independent of the height. At the outer boundary of the atmosphere, I'_ν, the radiation incident from outside is zero, and consequently

$$F_\nu = (I_\nu)_0.$$

Thus the net flux F_ν is measured by the residual intensity in the absorption line.

The actual amount of energy absorbed (and re-emitted) at any level depends on the fraction of the atoms at that level which are in the normal state; for only these atoms can absorb. Let n_1 be the number of atoms per unit volume, at any level, which are in the normal state, n_2 the number in the excited state. The number of absorptions per second will be proportional to n_1 and to the mean intensity of radiation to which they are subjected; the number of emissions * will be simply proportional to n_2. In the steady state the two must be equal, and it is found that

$$\frac{n_2}{n_1} = \frac{\frac{1}{2}(I_\nu + I'_\nu)}{\sigma},$$

where $\sigma = 2h\nu^3/c^2$. Now, though their difference remains constant from level to level, I_ν and I'_ν both increase inwards. This follows from the mechanism of formation of the absorption line; it can also be seen by noting that I'_ν is zero at the boundary, but at any internal point has a value which depends on the number of emitting atoms between the point in question and the boundary. It follows that n_2/n_1 increases inwards.

We now consider the mechanical equilibrium. At any level the upward momentum communicated per unit volume per second in virtue of absorptions of outward-flowing radiation will be

$$(\pi I_\nu \Delta\nu) a n_1/c,$$

where a is the atomic absorption coefficient for an atom in the normal state; the downward momentum is similarly

$$(\pi I'_\nu \Delta\nu) a n_1/c.$$

The resultant of these is $\pi(I_\nu - I'_\nu)\Delta\nu a n_1/c$, or

$$\pi F_\nu \Delta\nu a n_1/c.$$

The emissions make no contribution.† Since the total number of

* To avoid complicating the verbal arguments of this section, the effect of the "stimulated" emissions is ignored. These are taken into full account in the body of the paper. The formulæ quoted in this section are mostly approximations.

† This should not be misunderstood. The emissions at any given level cause no gain of momentum, but of course the total emission of the atmosphere to the layers beneath it involves an addition to the momentum calculated simply from the *apparent* absorption; in fact it doubles it. The contradiction is only apparent. At any given level a certain amount of momentum is absorbed from (say) the upward beam, but this exceeds that calculated from the apparent diminution of the beam, since the latter is also reinforced by emissions. We can say that emissions cause

atoms per unit volume is (n_1+n_2), we have for the equation of equilibrium

$$\frac{dp}{dx} = -mg(n_1+n_2) + \frac{a\pi F_\nu \Delta\nu n_1}{c},$$

p being the pressure, m the mass of an atom, and g the acceleration due to gravity; the co-ordinate x is measured positive outwards. Thus the ratio of the force arising from radiation pressure to the force arising from gravity, at any level, is

$$\frac{a\pi F_\nu \Delta\nu}{mgc} \cdot \frac{n_1}{n_1+n_2}.$$

In this, the first factor is constant; the second steadily decreases as we descend, since n_2/n_1 increases. The whole density-structure of the atmosphere will be found to depend on this circumstance. It should be contrasted with Eddington's theory of the *interior* of a star, where the force arising from radiation pressure is in a constant ratio to the force of gravity.

Additional equations are provided by the usual equations of transfer of radiation; they are found to reduce to

$$\frac{1}{a}\frac{dI_\nu}{dx} = \frac{1}{a}\frac{dI'_\nu}{dx} = -n_1 F_\nu.$$

From the three independent relations between I_ν and I'_ν, namely, the transfer equations and the equation expressing that the atoms are in a steady state, we can eliminate I_ν and I'_ν, obtaining an equation in n_2/n_1 strictly analogous to the equation giving the temperature-distribution in the thermodynamic theory of radiative equilibrium. Combining it with the equation of mechanical equilibrium (with some such assumption as $p=(n_1+n_2)k\mathrm{T}$), we obtain n_1 and n_2 separately. The density-distribution and the allocation amongst the stationary states at each height are thus known.

The character of the state of equilibrium defined by our system of equations will be found to depend intimately on F_ν, the residual intensity in the absorption line. In the absence of radiation, F_ν is zero, and the state is one of simple gravitational equilibrium. If F_ν is not zero, the gas is supported partly by radiation pressure, and any value of F_ν which does not make dp/dx positive* will define an atmosphere in equilibrium. There will, in fact, be a series of states of equilibrium, corresponding to increasing values of F_ν, in which an increasing fraction of the weight is supported by radiation pressure. But there is one limiting case which is of special interest. It is that in which, at great heights, the whole of the weight of the atom is supported by radiation pressure. It will correspond to the greatest possible value of F_ν.

no momentum change, but conceal the true absorption of momentum; or we can add to the apparent absorption of momentum an equal amount of recoil momentum associated with the total emission.

* This point, which is of importance in connection with Jeans' theory of planetary nebulæ, is considered in detail in § 18.

An atmosphere in which this limiting condition holds will be said to be "fully supported by radiation pressure." The terminology is not to be taken to imply that the whole mass of the atmosphere is supported by radiation pressure. In fact, at any finite height there will be a non-zero pressure gradient helping to support the atoms; for it follows from the foregoing that the fraction of the mass supported by the pressure gradient increases as we descend, even if it is zero initially (*i.e.* at great heights). The phrase is intended to denote simply an atmosphere in which the ratio of the mass supported by gas pressure to the mass supported by radiation pressure tends to zero at great heights. An atmosphere in which the limit of this ratio is not zero will be said to be "partially supported by radiation pressure."

In the general case, let $(1-\mu)$ be the fraction of gravity which is balanced by radiation pressure at great heights. We have then

$$mg(1-\mu)=\frac{a\pi F_\nu \Delta\nu}{c}\lim_{x\to\infty}\frac{n_1}{n_1+n_2},$$

and also

$$\lim_{x\to\infty}\frac{n_2}{n_1}=\frac{F_\nu}{2\sigma}.$$

It appears from these that n_2/n_1 is small compared with unity. Approximating, we find

$$F_\nu=\frac{cmg(1-\mu)}{\pi a\Delta\nu},$$

$$\lim\frac{n_2}{n_1}=\frac{cmg(1-\mu)}{2\pi\sigma a\Delta\nu}.$$

It is an important consequence of the theory of radiation that $a\Delta\nu$ is given in terms of τ, the average life of the atom in the excited state, by the formula

$$a\Delta\nu=\frac{h\nu}{4\pi\sigma\tau}.$$

Inserting the value $\sigma=2h\nu^3/c^2$, we find finally

$$F_\nu=\frac{8\nu^2\tau mg}{c}(1-\mu),$$

$$\lim\frac{n_2}{n_1}=\frac{2\tau cmg}{h\nu}(1-\mu).$$

The former of these formulæ for the case of an atmosphere fully supported by radiation pressure ($\mu=0$) is the relation found directly in my previous paper by a more pictorial method, and there used to afford an estimate of τ for a Ca^+ atom from the observed value of F_ν for the H and K lines in the solar spectrum. The important point to notice is that F_ν, the residual intensity in the line, is independent of $\Delta\nu$, the width of the absorption line, and independent of the effective temperature of the star. It is independent of such considerations as "the

number of atoms per unit area." (We shall see later, in fact, that given, for example, $\mu = 0$, it is F_ν which fixes the number of atoms supported per unit area, and not *vice versa.*) Through ν, τ, and m, F_ν depends on the atom concerned, and through g it depends on the star. For the rest, it depends on the parameter μ. We shall now see that consideration of the density-distribution allows us to deduce that for Ca^+ atoms in the solar chromosphere, μ must be very nearly zero, *i.e.* that the Ca^+ chromosphere constitutes an atmosphere fully supported by radiation pressure.

It is shown in Sections II. and III. that, when $\mu = 0$, the density at height x is given by an expression of the form

$$\rho = \frac{\text{const.}}{(x + x_0)^2},$$

where x_0 is the height of the "equivalent homogeneous atmosphere," reckoned from the level $x = 0$. On the other hand, when $\mu \neq 0$, except for extremely small values of μ the density is given approximately by

$$\rho = \text{const.} \times e^{-\mu m g x/(\text{const.})}.$$

It is practically that of an isothermal atmosphere in ordinary gravitational equilibrium with gravity reduced to μg. Thus, when $\mu = 0$, the rate of decrease of density outwards is extremely slow compared with the exponential decrease for $\mu \neq 0$. An atmosphere for which $\mu \neq 0$ is heavily condensed towards its lower layers; for $\mu = 0$, if we take a Ca^+ atmosphere under solar conditions, the density diminishes only by a factor of $\frac{1}{23}$ in 14,000 km. It is known that the theoretical exponential density-laws found by various writers are inadequate to account for the observed extent of the solar chromosphere. Observation and theory can be reconciled only if μ is zero or nearly zero.

The sudden change in the character of the density-distribution as μ approaches zero is one of the main results of the investigation. Further discussion may now be postponed until the formulæ have been derived.

I. *General Theory.*

3. *Assumptions.*—For precision, the assumptions made in the previous paper concerning the atoms of ionised calcium may be recapitulated. It is assumed that the high-level chromosphere consists of free electrons and singly ionised atoms of calcium. The latter, of mass m, are supposed to be effectively capable of two, and only two, stationary states, the 1σ or normal state and the 1π or excited state.* The pressure is so low that all the atoms are singly ionised, but second-stage ionisation is inappreciable. Though the pressure is low, we now take full account of the part played by the pressure gradient in supporting the atoms.

The analysis will be given for an atmosphere in which the level surfaces are parallel planes, with a constant value of the acceleration due

* It was shown that for the purposes of the investigation the $1\pi_1$ and $1\pi_2$ states of the Ca^+ atom may be regarded as forming a single 1π state.

to gravity. It therefore holds only for heights above the sun's surface so small that the effects of curvature may be neglected.* The height of the calcium chromosphere, if taken as 14,000 km., is only 2 per cent. of the sun's radius, so that neglect of curvature is justifiable.

4. *Radiation Theory.*—Let n_1, n_2 be the numbers of atoms per cm.^{3}, at a height x above some convenient reference point, in the 1σ state and the 1π state respectively. The states will be referred to for brevity as the states 1 and 2.

We introduce the Einstein probability coefficients $B_{1\to2}$, $A_{2\to1}$, $B_{2\to1}$, defined as follows. The probability that an atom in the state 1, exposed to isotropic radiation of specific intensity $I_\nu d\nu$, will absorb a quantum $h\nu$ during the interval dt is $B_{1\to2}I_\nu dt$. The probability that an atom in the state 2 will of itself emit a quantum $h\nu$ during dt is $A_{2\to1}dt$, and the additional probability that it will emit a quantum when exposed to isotropic ν-radiation is $B_{2\to1}I_\nu dt$. It is assumed that when the radiation is not isotropic, the probabilities corresponding to the element of radiation confined in an elementary cone $d\omega$ are obtained by replacing I_ν by $(I_\nu d\omega)/4\pi$. The probability coefficients are purely atomic quantities, independent of statistical variables such as the temperature, and independent of the incident radiation.

Einstein's relations † are

$$\frac{A_{2\to1}}{B_{1\to2}}=\frac{2h\nu^3}{c^2}, \qquad B_{2\to1}=B_{1\to2} \qquad . \quad . \quad (1), (2)$$

where c is the velocity of light. Einstein's relations are derived from a study of thermodynamic equilibrium. But since the coefficients are atomic quantities, the relations are permanent relations which will hold also in states which (as in the chromosphere) are not states of thermodynamic equilibrium. The utilisation of Einstein's relations for the atoms composing systems which are not in thermodynamic equilibrium is fundamental in the present investigation.

We proceed to connect $B_{1\to2}$ with the atomic absorption coefficient.‡ The finite breadth of an observed spectral line must be supposed to arise from two causes. In the first place, there will be an intrinsic breadth depending on the structure of the atom. In the second place, owing

* If the mass of the atmosphere is neglected, the equations can still be solved without trouble when curvature is taken into account. For all practical purposes the solution is in no wise different from that in the plane case. The only difficulty is one which occurs in the hydrostatics of the isothermal gas-sphere, namely, that ρ does not tend to zero but to a small positive constant at infinity. It does not seem worth while embarking on calculations which would take into account the mass of the atmosphere. *Cf.* Milne, *Camb. Phil. Trans.*, 22, 485, 1923, where a similar difficulty is discussed and the solution given in full.

† Einstein, *Phys. Zeit.*, 18, 121, 1917. An independent deduction in the present notation is given by Milne, *Phil. Mag.*, 47, 216, 1924. See also Planck, *Wärmestrahlung*, 5th edition (1923), p. 174, for the meaning of the coefficients from the point of view of Bohr's correspondence principle. In equations (1) and (2), the *a priori* weights of the stationary states are taken as equal. It is easy to revise the analysis so as to take account of unequal weights, if necessary.

‡ In the previous paper, a factor $d\nu$ depending on the breadth of the line was incorporated in the definitions of $B_{1\to2}$, $B_{2\to1}$. The present more refined procedure is an adaptation of that used by Tolman, *Phys. Rev.*, 23, 693, 1924.

to the Doppler effect of its own motion and to the random fields of neighbouring atoms, any particular atom at any particular instant will be capable of absorbing (or emitting) not its standard frequency (which we will denote temporarily by ν_0) but some neighbouring frequency ν. Arguments based on the correspondence principle suggest that the intrinsic width is negligible compared with the broadening arising from other causes, and we shall therefore neglect it. Let us now suppose that of the n_1 atoms in the state 1 capable of absorbing a frequency in the neighbourhood of ν_0, the number $n_1(\nu)d\nu$, owing to extraneous circumstances, will actually absorb radiation of frequency between ν and $\nu + d\nu$. Then the number of absorptions of this radiation, per unit volume per second, is

$$n_1(\nu)d\nu B_{1\to 2}I_\nu,$$

and the amount of radiation absorbed is this multiplied by $h\nu$. The amount of radiation incident is $4\pi I_\nu d\nu$. Hence the apparent atomic absorption coefficient $\alpha(\nu)$ for the atoms in state 1 is given by

$$\alpha(\nu) = \frac{B_{1\to 2}h\nu I_\nu n_1(\nu)d\nu}{4\pi I_\nu d\nu n_1} = \frac{B_{1\to 2}h\nu n_1(\nu)}{4\pi n_1}.$$

This is the value which would be obtained by measurement of the actual contour of an absorption line produced, for example, by illuminating a slab of the gas (containing a known number of atoms in the state 1) with radiation of known intensity.

We now assume that $B_{1\to 2}$ is the same for all atoms in the state 1, independent of the extraneous circumstances. We assume also that the function $n_1(\nu)$ changes rapidly in the neighbourhood of $\nu = \nu_0$, becoming zero for comparatively small values of $\nu - \nu_0$. Integrating through the absorption line, and recalling that

$$\int_0^\infty n_1(\nu)d\nu = n_1,$$

we have

$$\int_0^\infty \alpha(\nu)d\nu = \frac{B_{1\to 2}h\nu_0}{4\pi} \quad . \quad . \quad . \quad . \quad (3)$$

very closely. We can now omit the suffix 0. It is convenient to write the integral on the left as $\alpha\Delta\nu$, where $\Delta\nu$ has been defined in some way as the "width" of the line, and α is the corresponding mean value of the atomic absorption coefficient for atoms in the state 1. Then

$$\alpha\Delta\nu = \frac{B_{1\to 2}h\nu}{4\pi} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The mean life τ of an atom in the state 2, undisturbed by external radiation, is given by

$$\tau = 1/A_{2\to 1} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We shall write

$$\sigma = 2h\nu^3/c^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Combining (4), (5), and (6) with the Einstein relation (1), we have the important relation

$$\alpha\Delta\nu = \frac{h\nu}{4\pi\tau\sigma} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In § 16 we shall return to the detailed relation (3) in order to verify that the calculated value of the residual intensity, as determined by the analysis of § 8, refers to the centre of the observed absorption line in the solar spectrum. In the meantime, we shall assume for simplicity that the absorption line is sharply bounded and has an *actual* width $\Delta\nu$, within which the residual intensity is constant.

5. *The Equations of Radiative Equilibrium.*—Let $I_\nu d\nu$ be the intensity of ν-radiation at the height x, in a direction making an angle θ with the outward vertical. The number of absorptions per unit volume at height x, in time dt, is

$$n_1 B_{1\to 2} dt \int I_\nu \frac{d\omega}{4\pi}.$$

The number of emissions, including the stimulated emissions, is

$$n_2 dt\left(A_{2\to 1} + B_{2\to 1}\int I_\nu \frac{d\omega}{4\pi}\right).$$

Equating the two, and using Einstein's relations, we have as the condition of radiative equilibrium

$$n_1\int_0^\pi I_\nu \sin\theta d\theta = n_2\int_0^\pi (\sigma + I_\nu)\sin\theta d\theta \quad . \quad . \quad . \quad (8)$$

To obtain the equation of transfer of radiation, we consider a thin slab of the gas, of area dS and thickness dx, bounded by horizontal planes. The number of quanta of ν-radiation entering the slab in directions confined within the solid angle $d\omega$ is

$$\frac{I_\nu \Delta\nu \cos\theta d\omega dS}{h\nu},$$

and the number leaving in the same directions is

$$\frac{(I_\nu + dI_\nu)\Delta\nu \cos\theta d\omega dS}{h\nu}.$$

The excess of the latter number over the former must be equal to the excess of emissions of ν-quanta in the directions included in $d\omega$, inside the slab, over the number of similar absorptions. The number of such emissions is *

$$n_2 dx dS(A_{2\to 1} + B_{2\to 1} I_\nu)\frac{d\omega}{4\pi},$$

* It is here assumed that the stimulated radiation consists in the emission of a quantum in the same direction as a stimulating quantum. *Cf.* Bohr, *Zeit. für Phys.*, 20, 145, 1923. No material difference results, for our purpose, if the stimulated emission occurs in a random direction, or indeed if the stimulated emission is neglected entirely.

and the number of such absorptions is

$$n_1 dx d\mathrm{S} \mathrm{B}_{1\to 2} \mathrm{I}_\nu \frac{d\omega}{4\pi}.$$

Hence, after a little simplification, on using (1), (2), and (4) we find

$$\frac{\cos\theta}{a}\frac{d\mathrm{I}_\nu}{dx} = -n_1\mathrm{I}_\nu + n_2(\sigma + \mathrm{I}_\nu) \quad . \quad . \quad . \quad (9)$$

Multiplying (9) by $\sin\theta d\theta$ and integrating from 0 to π, we find on using (8)

$$\frac{d}{dx}\int_0^\pi \mathrm{I}_\nu \cos\theta \sin\theta d\theta = 0.$$

This shows that the net outward flux, in the line, is the same across all levels. If $\pi \mathrm{F}_\nu \Delta\nu$ is the constant net flux, then

$$\pi \mathrm{F}_\nu = 2\pi \int_0^\pi \mathrm{I}_\nu \cos\theta \sin\theta d\theta \quad . \quad . \quad . \quad (10)$$

The quantity F_ν is the observed residual intensity in the absorption line, averaged over the sun's disc.*

6. *The Equation of Mechanical Equilibrium.*—In time dt, the upward momentum communicated to a slab of area $d\mathrm{S}$ and thickness dx by the absorption of radiation is

$$n_1 d\mathrm{S} dx \mathrm{B}_{1\to 2} dt \frac{h\nu}{c}\int \mathrm{I}_\nu \cos\theta \frac{d\omega}{4\pi}$$
$$= \frac{n_1 a\pi \mathrm{F}_\nu \Delta\nu}{c} dx d\mathrm{S} dt,$$

on using (4) and (10). Unstimulated emissions will be distributed in direction at random, and on the average the recoil momentum due to them will be zero. The recoil momentum in the upward direction due to the stimulated emissions is

$$-n_2 d\mathrm{S} dx \mathrm{B}_{2\to 1} dt \frac{h\nu}{c}\int \mathrm{I}_\nu \cos\theta \frac{d\omega}{4\pi}$$
$$= -\frac{n_2 a\pi \mathrm{F}_\nu \Delta\nu}{c} dx d\mathrm{S} dt.$$

The downward momentum acquired under gravity in time t is

$$(n_1 + n_2) dx d\mathrm{S} m g dt.$$

Hence the equation of equilibrium is

$$\frac{dp}{dx} = -(n_1 + n_2)mg + \frac{a\pi \mathrm{F}_\nu \Delta\nu}{c}(n_1 - n_2) \quad . \quad . \quad (11)$$

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* *Cf.* Milne, *Phil. Trans.*, **223**, A, 212 (footnote), 1922.

where p is the total pressure. Since the atoms are all singly ionised, p is twice the partial pressure of the Ca^+ atoms. In deducing this equation it has been assumed that the radiation exerts no pressure on free electrons. The mass m in (11) is strictly the sum of the masses of a Ca^+ atom and an electron. The system as a whole, ions with electrons, is then in equilibrium, and though radiation pressure acts only on the ions, the electrostatic forces between the ions and the electrons will prevent any appreciable macroscopic separation of charge.*

For simplicity we shall now assume that the chromosphere is at some uniform temperature T_0. The temperature gradients in the chromosphere are probably small. But the assumption is not essential; the procedure would be the same whatever the temperature distribution. However, the assumption allows integration of the equation to be effected very simply. We have then †

$$p = 2(n_1 + n_2)kT_0 \quad . \quad . \quad . \quad . \quad (12)$$

whence, inserting in (11),

$$2kT_0 \frac{d}{dx} \log(n_1 + n_2) = -mg + \frac{\alpha\pi F_\nu \Delta\nu}{c} \frac{n_1 - n_2}{n_1 + n_2} \quad . \quad . \quad (13)$$

7. *Approximate Treatment of the Radiation Equations.*—Equations (8), (9), and (13) form a system for determining n_1 and n_2 as functions of x, and I_ν as a function of x and θ. To simplify the solution, we shall now replace I_ν by mean values. Write

$$I_\nu(\theta) = \bar{I}_\nu \qquad (0 \leqslant \theta < \tfrac{1}{2}\pi),$$
$$I_\nu(\theta) = \bar{I}'_\nu \qquad (\tfrac{1}{2}\pi < \theta \leqslant \pi),$$

so that $\bar{I}_\nu$, $\bar{I}'_\nu$ are the mean intensities of the outward and inward streams of radiation. For convenience we shall now omit the bars and write the mean intensities simply as I_ν, I'_ν. Equations (8) and (10) now approximate to

$$(n_1 - n_2)(I_\nu + I'_\nu) = 2n_2\sigma \quad . \quad . \quad . \quad . \quad (14)$$

$$I_\nu - I'_\nu = F_\nu \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Multiplying (9) by $\sin\theta d\theta$ and integrating in turn from 0 to $\frac{1}{2}\pi$ and from $\frac{1}{2}\pi$ to π, we find, on using (14) and (15),

$$\frac{dI_\nu}{dx} = \frac{dI'_\nu}{dx} = -\alpha(n_1 - n_2)F_\nu \quad . \quad . \quad . \quad (16)$$

The approximation here made is the Schuster-Schwarzschild approximation for radiative equilibrium.‡ It would not be difficult to carry the approximation further, *e.g.* by using a cosine approximation for I_ν, but it seems not worth while at this stage to introduce the whole machinery of the analysis of radiative equilibrium. The error introduced by the present approximation will be referred to in § 17.

* Pannekoek, *B.A.N.*, **19**, 110, 1922; Rosseland, *Monthly Notices*, **84**, 720, 1924.

† For a non-ionised gas, the factor 2 would be replaced by unity.

‡ Schwarzschild, *Gött. Nach.*, 1906, p. 41. For an analysis of the errors, *v.* Schwarzschild, *Berlin Sitz.*, 1914, p. 1183; Milne, *Monthly Notices*, **81**, 361, 1921, and *Phil. Trans.*, **223**, A, 230, 1922.

II. *Atmosphere fully supported by Radiation Pressure.*

8. *The Boundary Condition.*—We now proceed to solve the equations. The analysis for an atmosphere fully supported by radiation pressure differs from that for an atmosphere partially supported, and will be given separately.

For an atmosphere fully supported by radiation pressure, the right-hand side of (13) must vanish at great heights, and we have

$$mg = \frac{a\pi F_\nu \Delta\nu}{c} \lim_{x\to\infty} \frac{n_1 - n_2}{n_1 + n_2} \quad . \quad . \quad . \quad . \quad (17)$$

The radiation incident on the atmosphere from outside is zero. Consequently, at great heights $I'_\nu = 0$ and $I_\nu = F_\nu$. Hence from (14)

$$\frac{F_\nu}{2\sigma} = \lim_{x\to\infty} \frac{n_2}{n_1 - n_2} \quad . \quad . \quad . \quad . \quad (18)$$

Equations (17) and (18) determine F_ν and the limit of n_2/n_1. Solving, we find

$$F_\nu = \frac{\sigma}{s-1} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\lim_{x\to\infty} \frac{n_2}{n_1 - n_2} = \frac{\frac{1}{2}}{s-1} \quad . \quad . \quad . \quad . \quad (20)$$

where

$$s = \frac{\pi\sigma a \Delta\nu}{cmg} = \frac{h\nu}{4\tau cmg} \quad . \quad . \quad . \quad . \quad (21)$$

Thus the residual intensity in the line, F_ν, has a perfectly definite value, involving the atomic constant τ.

If the residual intensity in the line is some fraction r of the intensity in the neighbouring continuous spectrum, then, assuming that the continuous spectrum is that of a black body, at temperature T, we have

$$F_\nu = r\,\frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT} - 1} \quad . \quad . \quad . \quad . \quad (22)$$

Combining with (19), since $\sigma = 2h\nu^3/c^2$, we have

$$r = \frac{e^{h\nu/kT} - 1}{s-1} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If r is known from observation, (23) affords a determination of s and hence of τ.

It is readily found that s is large compared with unity. Hence, replacing $s-1$ by s, we have approximately

$$F_\nu = \frac{4\sigma\tau cmg}{h\nu} = \frac{8\nu^2\tau mg}{c} \quad . \quad . \quad . \quad . \quad (24)$$

$$r = \frac{4\tau mgc}{h\nu}\left(e^{h\nu/kT} - 1\right) \quad . \quad . \quad . \quad . \quad (25)$$

$$\lim \frac{n_2}{n_1} = \frac{2\tau cmg}{h\nu} \quad . \quad . \quad . \quad . \quad (26)$$

If n_2 is neglected in comparison with n_1 in (17), formula (24) follows immediately. In my previous paper, where also (24) was found, the occurrence of the atomic absorption coefficient a, of the interval $\Delta\nu$, and of the concentrations n_1 and n_2 was avoided by considering a single atom and introducing the average time τ' which an atom spends in the normal state. The present more conventional method confirms the result of the earlier. It is readily verified that at any height

$$n_2/n_1 = \tau/\tau'.$$

It is curious that F_ν, according to (24), is independent of Planck's constant h.*

Assuming the high-level calcium chromosphere constitutes an atmosphere fully supported by radiation pressure, the adopted values $g = 2{\cdot}73 \times 10^4$ cm. sec.$^{-2}$, $T = 6000°$, $m = 40 \times 1{\cdot}65 \times 10^{-24}$ grams, $\lambda = 3950$ Å., $r = 0{\cdot}11$, inserted in (25), give $\tau = 0{\cdot}60 \times 10^{-8}$ sec. We find also $s = 3{\cdot}85 \times 10^3$, lim $(n_2/n_1) = 1{\cdot}30 \times 10^{-4}$.

9. *Solution of the Equations.*—For brevity write

$$n = n_1 - n_2.$$

From (14) and (15) we find

$$2I_\nu - F_\nu = 2\sigma n_2/n \quad . \quad . \quad . \quad . \quad (27)$$

Inserting for I_ν in (16) and using the value of F_ν given by (19), we have

$$\frac{d}{dx}\left(\frac{n_2}{n}\right) = -\frac{na}{s-1} \quad . \quad . \quad . \quad . \quad (28)$$

We shall find the solution occurs most simply in terms of the ratio n_2/n, and we therefore introduce a new variable η, defined by

$$\frac{n_2}{n} = \frac{\frac{1}{2}(1+\eta)}{s-1} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

so that, in virtue of (20), $\eta \to 0$ as $x \to \infty$. Equation (28) now becomes

$$\tfrac{1}{2}\frac{d\eta}{dx} = -na,$$

which may also be written in the form

$$-(n+2n_2)a = \tfrac{1}{2}\left(1 + \frac{2n_2}{n}\right)\frac{d\eta}{dx}$$

$$= \frac{\frac{1}{2}(s+\eta)}{s-1}\frac{d\eta}{dx} \quad . \quad . \quad . \quad . \quad (30)$$

In terms of η, equation (13) will be found to reduce to

$$\frac{2kT_0}{mg}\frac{d}{dx}\log(n+2n_2) = \frac{n}{n+2n_2}\lim_{x\to\infty}\left(\frac{n+2n_2}{n}\right) - 1$$

$$= -\frac{\eta}{s+\eta} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

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* Presumably a similar formula could be deduced on the basis of the classical theory of radiation. The quantity τ would be expected to be replaced by a factor involving the electromagnetic damping constant.

Equations (30) and (31) form a pair of differential equations for η and $(n+2n_2)$. Multiplying them together, we observe that each side of the resulting equation forms a perfect differential, and integrating we find

$$\frac{2akT_0}{mg}(n+2n_2)=\frac{\frac{1}{4}\eta^2}{s-1} \quad . \quad . \quad . \quad . \quad (32)$$

The constant of integration is zero, since as $x\to\infty$ both $\eta\to 0$ and $(n+2n_2)\to 0$. Equation (32) gives the density $\rho=m(n+2n_2)$ in terms of η.

To express η in terms of x, insert the value of $(n+2n_2)$ from (32) in (30), and integrate. We find

$$x+x_0=\frac{4kT_0}{mg}\left(\frac{s}{\eta}-\log\eta\right) \quad . \quad . \quad . \quad . \quad (33)$$

where x_0 is some constant.

10. *Mathematical Results.*—We now collect our main results, with the addition of formulæ for I_ν and I'_ν, which are given immediately by (14) and (15). It is convenient to return to the variables n_1 and n_2, and to use the mean mass absorption coefficient k_ν given by $k_\nu=\alpha/m$. The formulæ are:

$$\frac{n_2}{n_1-n_2}=\frac{\frac{1}{2}(1+\eta)}{s-1} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$n_1+n_2=\frac{g}{8k_\nu kT_0(s-1)}\eta^2 \quad . \quad . \quad . \quad . \quad (35)$$

$$\rho=\frac{mg}{8k_\nu kT_0(s-1)}\eta^2, \qquad p=\frac{g}{4k_\nu(s-1)}\eta^2 \quad . \quad (36), (37)$$

$$I_\nu=\frac{\sigma(1+\frac{1}{2}\eta)}{s-1}, \qquad I'_\nu=\frac{\frac{1}{2}\sigma\eta}{s-1}, \qquad F_\nu=\frac{\sigma}{s-1} \quad (38), (39), (40)$$

$$x+x_0=\frac{4kT_0}{mg}\left(\frac{s}{\eta}-\log\eta\right) \quad . \quad . \quad . \quad . \quad (41)$$

The mass per unit area above the level η is given by

$$\int_{\eta=0}^{\eta}\rho dx=\frac{\eta(\eta+2s)}{4k_\nu(s-1)} \quad . \quad . \quad . \quad . \quad (42)$$

In (42) it can be verified that the term in η represents the mass supported by radiation pressure, the term in η^2 the mass supported by gas pressure.

In deducing (34) to (43), no approximations have been made except in the simplification of the equations of radiative equilibrium. The only undetermined constant is x_0. It may be recalled that σ is $2h\nu^3/c^2$, s is $h\nu/4\tau cmg$, and that $1/k_\nu$ is $4\pi m\sigma\tau\Delta\nu/h\nu=8\pi\nu^2\tau m\Delta\nu/c^2$.

11. *The Density Law.*—Elimination of η between (36) and (41) gives the law of variation of density with height. To obtain this in a convenient form, we shall now approximate, having regard to the numerical values. With the value $s=3{\cdot}85\times10^3$ found in § 8, we can

replace $s-1$ by s without sensible error. Further, in (41) the term in $\log \eta$ is always negligible compared with s/η for the range of values we shall require to consider. Neglecting it, we find on elimination of η that

$$\rho = \frac{2kT_0}{mg} \cdot \frac{s}{k_\nu} \cdot \frac{1}{(x+x_0)^2}.$$

But

$$s/k_\nu = \pi\sigma\Delta\nu/cg \quad . \quad . \quad . \quad . \quad (43)$$

Hence

$$\rho = \frac{2\pi kT_0\sigma\Delta\nu}{mg^2c(x+x_0)^2} \quad . \quad . \quad . \quad . \quad (44)$$

and the mass above any level is

$$\int_x^\infty \rho dx = \frac{2\pi kT_0\sigma\Delta\nu}{mg^2c(x+x_0)} \quad . \quad . \quad . \quad . \quad (45)$$

The law of density given by (44) offers several points of interest. According to this law, the density at any point is inversely proportional to the square of the height of the point above a certain fixed level, which is at a distance x_0 below the level $x=0$. It has been pointed out by various writers that exponential laws give far too rapid a rate of decrease of density outwards to be reconciled with observation. The rate of decrease given by (44) is extremely slow in comparison.

To determine the constant x_0 some additional assumption is required. For example, it could be determined if the density at some particular height could be fixed. However, we adopt a different procedure. The analysis so far given refers solely to the high-level calcium vapour, where all the calcium atoms are singly ionised. It will not necessarily hold in the lower chromospheric levels, where an appreciable fraction of the Ca atoms are neutral and where there is admixture with other atoms; still less will it be expected to hold in the reversing layer. In these regions the state of affairs must be comparatively complicated, and it is not at present clear what simplifying assumptions are legitimate in attempting to represent the conditions analytically. But the analysis of the present paper would hold rigorously, at all depths, in an atmosphere composed solely of Ca^+ atoms in the 1σ and 1π states and an equal number of free electrons; for full account has been taken both of radiation pressure and of gas pressure. As we penetrate deeper into such an hypothetical atmosphere, the mean intensity I_ν of the outward radiation steadily increases, and we shall eventually reach a depth where I_ν attains a value equal to the observed photospheric radiation in the neighbourhood of ν. If the equilibrium of the Ca^+ atmosphere could be considered entirely on its own, it would be reasonable to suppose that the depth from which the general photospheric radiation was coming was in fact the depth at which I_ν for the H and K radiation, calculated from the equilibrium of the Ca^+ atmosphere, attained the photospheric value. This may not actually be the case, for simple chromospheric equilibrium of the type here considered probably ceases, as we descend, before the layers are reached in which the ordinary

strong absorption lines are formed. However, it seems reasonable to assume tentatively that for the purpose of determining the order of magnitude of x_0 we may take the photospheric depth to be fixed by the condition just stated. We therefore assume that at $x=0$ (which we take to be the level corresponding to the apparent limb) I_ν has the photospheric value $\sigma/(e^{h\nu/kT}-1)$.

Let η_0 be the value of η at $x=0$. Using the value of I_ν given by (38), we find

$$\frac{\sigma(1+\frac{1}{2}\eta_0)}{s-1}=\frac{\sigma}{e^{h\nu/kT}-1},$$

or, using (23),

$$\eta_0=2\left(\frac{1}{r}-1\right) \quad . \quad . \quad . \quad . \quad . \quad (46)$$

For our adopted value $r=0{\cdot}11$, this gives $\eta_0=16$. Thus n_2/n_1 increases 17-fold between great heights and $x=0$. The magnitude of η_0 justifies our neglect of $\log\eta$ compared with s/η in (41), since only values of η between 0 and 16 come into consideration. Finally, from (41), putting $x=0$,

$$x_0=\frac{4kT_0s}{mg\eta_0} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Using (46) and (23), this may be put in the form

$$x_0=\frac{2kT_0}{mg}\cdot\frac{e^{h\nu/kT}-1+r}{1-r} \quad . \quad . \quad . \quad . \quad (48)$$

The total mass per unit area above the level $x=0$ is then found to be

$$\int_0^\infty \rho dx=\frac{\pi\sigma\Delta\nu(1-r)}{cg(e^{h\nu/kT}-1+r)} \quad . \quad . \quad . \quad . \quad (49)$$

For the sun, let us adopt $T_0=5000°$. This may be rather high for the temperature of the upper chromosphere, but little depends on the exact value. We then find $x_0=3{\cdot}66\times10^8$ cm., or 3660 km. From the limb to a height of 14,000 km. the density is therefore reduced in the ratio

$$\frac{(3{\cdot}66)^2}{(14+3{\cdot}66)^2}=\frac{1}{23{\cdot}3}.$$

With x_0 determined, the density and pressure at all heights are determined. For definiteness let us take $\Delta\nu$ as corresponding to a line-width $\Delta\lambda=1$ Å. For $x=0$, the formulæ previously found now give

$$\rho_0=2{\cdot}68\times10^{-17}\text{ gram. cm.}^{-3}, \qquad p_0=0{\cdot}555\times10^{-6}\text{ dyne cm.}^{-2},$$

$$(n_1+n_2)_0=4{\cdot}05\times10^5\text{ cm.}^{-3},$$

$$\int_0^\infty \rho dx=0{\cdot}978\times10^{-8}\text{ gram. cm.}^{-2}, \qquad \int_0^\infty (n_1+n_2)dx=1{\cdot}49\times10^{14}\text{ cm.}^{-2}.$$

For any other value of $\Delta\nu$, these figures must be altered proportionally. The mean free path corresponding to $4{\cdot}05 \times 10^5$ atoms per cm.^{3}, assuming an atomic diameter of 3×10^{-8} cm., is 6200 km., which at the mean speed of agitation at 5000° is described in 3500 sec., or about one hour. Though we have taken gas pressure into account, collisions are rare even at the base of the chromosphere. The ratio of the total mass (above $x=0$) supported by gas pressure to that supported by radiation pressure is only $\eta_0/2s = 2{\cdot}1 \times 10^{-3}$.

The quantity x_0 may be regarded as the height of the "equivalent homogeneous atmosphere" above the level $x=0$. Comparison of (45) and (44) shows at once that

$$x_0 = \frac{1}{\rho_0}\int_0^\infty \rho dx.$$

From this point of view, the value $x_0 = 3660$ km. for the calcium chromosphere seems very reasonable.

12. *Discussion of the Density Law for an Atmosphere fully supported by Radiation Pressure.*—The procedure adopted to determine x_0 may be open to question as a valid method for application to the sun. But it is easy to construct a model for which it has a perfectly definite significance. Let us consider a hypothetical Ca^+ atmosphere, bounded below by a fixed radiating surface with a black-body temperature T. We will suppose the amount of calcium per unit area to be adjusted until at high levels radiation pressure and gravity just balance; this is clearly possible, provided T is high enough, for if there is too little calcium, the atoms will be repelled by radiation pressure, and if there is too much, only a fraction of gravity will be balanced by radiation pressure at great heights. We have then an atmosphere fully supported by radiation pressure, and our analysis should describe it accurately. It will yield a continuous spectrum crossed only by the H and K lines, in which the residual intensity will be $F_\nu = 8\nu^2\tau mg/c$, which is independent of T or $\Delta\nu$. The mass per unit area must be just such as to reduce the residual intensity to this value; it is given by (49), where r is given by (25).

With $g = 2{\cdot}73 \times 10^4$ and $T = 6000°$, an adopted value $\tau = 0{\cdot}6 \times 10^{-8}$ sec. gives an intensity ratio $r = 0{\cdot}11$. Smaller values of τ give smaller intensity ratios. Since r must be less than unity, for a "photospheric temperature" of 6000° an atmosphere of Ca^+ atoms fully supported by radiation pressure is impossible (according to (25)) unless $\tau < 5{\cdot}4 \times 10^{-8}$ sec. Or again, with $\tau = 0{\cdot}6 \times 10^{-8}$ sec. a "photospheric temperature" of less than 4400° is incapable of yielding an atmosphere of Ca^+ atoms fully supported by radiation pressure.

The limiting value of n_2/n_1 at high levels is $1{\cdot}30 \times 10^{-4}$, which is the same as would exist in thermodynamic equilibrium at 4060°. There is, however, no presumption that the gas at high levels is, in fact, at a temperature of 4060°, if by "temperature" we mean the temperature corresponding to the mean velocity of the atoms.

The results so far enumerated in this section are entirely independent of the actual temperature distribution in the atmosphere. If we now assume an actual temperature of 5000°, we find that with $\tau = 0{\cdot}6 \times 10^{-8}$

sec., $T = 6000°$, $\Delta\lambda = 1$ Å., there must be $1{\cdot}49 \times 10^{14}$ atoms per unit area, corresponding to a pressure at the base of about $\frac{1}{2} \times 10^{-12}$ atmos. In calculating the distribution of density, gas-pressure gradients have been taken into account, but of the total mass supported only the fraction $\eta_0/2s = 2{\cdot}1 \times 10^{-3}$ is supported by gas pressure. The gas is supported almost entirely by radiation pressure. Thus with $\Delta\lambda = 1$ Å., $1{\cdot}49 \times 10^{14}$ atoms per cm.^{2} are supported by radiation pressure and produce an absorption line in which the residual intensity is 0·11 of the intensity in the neighbouring continuous spectrum. Changes in $\Delta\lambda$ change the number of atoms per cm.^{2}, but automatically provide for their being supported by radiation pressure; the residual intensity in the line remains the same.

These conclusions differ radically from some recently arrived at by Stewart.* Calculating on the basis of the classical theory of radiation, he estimated that 3×10^{16} atoms of sodium per cm.^{2} were required to produce just observable D-lines under solar conditions, taking them as 0·6 Å. wide and assuming an intensity ratio of $\frac{1}{2}$. He showed that this was about 500 times the mass that could be supported by radiation pressure, even if all the D-radiation were absorbed. The calculations of the present paper show no such 500-fold discrepancy between the mass required to produce the residual intensities observed in the centre of the H and K absorption lines and the mass that can be supported by radiation pressure. Mass sufficient to give an intensity ratio of $\frac{1}{9}$ can be supported by radiation pressure if $\tau = 0{\cdot}6 \times 10^{-8}$ sec. It is true that this value of τ is deduced on the *assumption* that the high-level atoms are just in equilibrium under radiation pressure, but any serious discrepancy would have shown itself by giving an absurd value for τ. If the deduced value of τ had come out absurdly small, as compared with the results of laboratory determinations or theoretical calculations on the structure of the atom, it would have meant that the line was too intense for the residual intensity to be strong enough to support high-level atoms by radiation pressure; if it had come out absurdly large, it would have meant that F_ν was larger than necessary to support the atoms against gravity, *i.e.* that atoms were being steadily expelled by radiation pressure. That the balance must be very nearly exact will appear from the investigation of the next section.

It must be emphasised that there is a radical difference between Stewart's procedure and that of this paper. Stewart began by calculating how many atoms were required to produce an absorption line of given intensity (assuming the atoms absorbed classically). In this paper we begin by determining the residual intensity which is just sufficient to support high-level atoms against gravity. This is determined in absolute units; it involves the time τ. We then work inwards from this value until we have traversed sufficient atoms for the outward intensity to equal the photospheric intensity. This is then equal to the total number actually supported by radiation pressure, apart from a small correction due to gas pressure.

* *Astrophys. Journ.*, **49**, 30, 1924.

III. *Atmosphere partially supported by Radiation Pressure.*

13. *Mathematical Results.*—We now investigate the nature of the state of equilibrium when the fraction $(1-\mu)$ of gravity is balanced by radiation pressure at great heights, where $0<\mu<1$.

The boundary condition (17) is replaced by

$$mg(1-\mu)=\frac{a\pi F.\Delta\nu}{c}\lim\frac{n_1-n_2}{n_1+n_2} \quad . \quad . \quad . \quad (50)$$

Condition (18) is unaltered. We then find

$$F_\nu=\frac{\sigma}{s'-1} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$\lim\frac{n_2}{n_1-n_2}=\frac{\frac{1}{2}}{s'-1} \quad . \quad . \quad . \quad . \quad (52)$$

where

$$s'=\frac{s}{1-\mu}=\frac{h\nu}{4\tau cmg(1-\mu)} \quad . \quad . \quad . \quad . \quad (53)$$

Hence

$$r=\frac{e^{h\nu/kT}-1}{s'-1} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

It will be found that the solution can be carried out as before. Introducing a parameter η' defined by

$$\frac{n_2}{n_1-n_2}=\frac{\frac{1}{2}(1+\eta')}{s'-1} \quad . \quad . \quad . \quad . \quad (55)$$

we find

$$n_1+n_2=\frac{g}{8k_\nu kT_0(s'-1)}\eta'(\eta'+2\mu s') \quad . \quad . \quad . \quad (56)$$

$$\rho=\frac{mg}{8k_\nu kT_0(s'-1)}\eta'(\eta'+2\mu s'), \qquad p=\frac{g}{4k_\nu(s-1)}\eta'(\eta'+2\mu s') \quad (57), (58)$$

$$I_\nu=\frac{\sigma(1+\frac{1}{2}\eta')}{s'-1}, \qquad I'_\nu=\frac{\frac{1}{2}\sigma\eta'}{s'-1} \quad . \quad . \quad (59), (60)$$

$$x+x_0=\frac{4kT_0}{mg}\left[\frac{1-2\mu}{2\mu}\log(\eta'+2\mu s')-\frac{1}{2\mu}\log\eta'\right] . \quad . \quad (61)$$

$$\int_{\eta'=0}^{\eta'}\rho dx=\frac{\eta'(\eta'+2s')}{4k_\nu(s'-1)} \quad . \quad . \quad . \quad . \quad (62)$$

14. *The Density Law.* — Combination of (57) and (61) gives a density law in marked contrast to that for $\mu=0$. A change in μ from zero to even a small positive value causes an enormous change in the character of the distribution of density. The question will be approached in two ways: we will first examine the new distribution of density by itself; secondly, examine the change in density at any given height.

As before, we measure x from the level at which I_ν is equal to the photospheric intensity. Hence, as before,

$$\eta'_0 = 2\left(\frac{1}{r} - 1\right) \quad . \quad . \quad . \quad . \quad (63)$$

If we consider r as given by the observed intensity ratio, η'_0 has the value previously found for η_0; on the other hand, if τ is given, r is less than before, approximately in the ratio $1-\mu : 1$, but if μ is small η'_0 will then differ only slightly from η_0. This shows that, throughout its range, η' is fairly small compared with $2\mu s'$, unless μ is exceedingly small. For example, if $\mu = 0{\cdot}01$, $2\mu s' = 77$. Hence in (57) we can neglect η' compared with $2\mu s'$, and in (61) we can replace $\log(\eta' + 2\mu s')$ by $\log 2\mu s'$. Eliminating η' from the approximate formulæ thus obtained, we find approximately

$$\rho \propto e^{-\mu mgx/2kT_0} \quad . \quad . \quad . \quad . \quad (64)$$

provided, say, $\mu \geqslant 0{\cdot}01$.

The distribution of density is therefore practically that of an atmosphere in gravitational equilibrium with the value of gravity reduced to μg. The value of $mg/2kT_0$ is $1{\cdot}31 \times 10^{-6}$. For a height of 14,000 km. we have $mgx/2kT_0 = 1830$. Hence for $\mu = 0{\cdot}01$ the density diminishes by a factor of $e^{-18{\cdot}3} = 10^{-8}$ in a change of height of 14,000 km.; for $\mu = 0{\cdot}1$ the factor is 10^{-80}. For $\mu = 0$ we saw that the density diminished by a factor of $\frac{1}{2{\cdot}3}$ in the same range of height. Thus if the atoms at high levels are entirely supported by radiation pressure, the density at the top of the chromosphere is $\frac{1}{2{\cdot}3}$ of its value at the base; but if even one-hundredth of the weight of an atom, at high levels, is uncompensated by radiation pressure and has to be borne by the pressure gradient, the density at the top of the chromosphere is reduced to one hundred-millionth of its value at the base.

The change in actual density at a given height, caused by a change in μ from zero to a small positive value, is perhaps even more striking. Take $\mu = 0{\cdot}01$. Then the same observed value $r = 0{\cdot}11$, inserted in (54), will give a numerical value for s' equal to that previously found for s, $3{\cdot}85 \times 10^3$. The resulting value of τ will be greater by 1 per cent. Further, $\eta'_0 = \eta_0 = 16$. From (61), since $x = 0$ when $\eta' = \eta'_0$, we have

$$\frac{mgx}{4kT_0} = \frac{1-2\mu}{2\mu} \log \frac{\eta' + 2\mu s'}{\eta'_0 + 2\mu s'} - \frac{1}{2\mu} \log \frac{\eta'}{\eta'_0} \quad . \quad . \quad (65)$$

For a height x of 14,000 km., this will be found to give $\eta' = 1{\cdot}39 \times 10^{-7}$. When $\mu = 0$, the value of η at 14,000 km. is $16 \times 3{\cdot}66/(14 + 3{\cdot}66) = 3{\cdot}32$. Consequently the ratio of the densities at 14,000 km. in the two cases is

$$\frac{\eta'(\eta' + 2\mu s')}{\eta^2} = 0{\cdot}97 \times 10^{-6}.$$

On the other hand, the ratio of the densities at $x = 0$ is

$$\frac{\eta'_0(\eta'_0 + 2\mu s')}{\eta_0^2} = 5{\cdot}8.$$

We see that if radiation pressure is supposed to balance 99 per cent. of gravity at great heights instead of 100 per cent., the intensity in the line being supposed the same, the density at $x=0$ is increased to six times its value, but the density at 14,000 km. is reduced to one-millionth. For $\mu=0{\cdot}1$ the ratios are 49 and 10^{-77}. The total mass supported is exactly the same in all three cases, by (62), η'_0 and s' being equal to η_0 and s if the residual intensity in the absorption line is supposed given.

The ratios are practically unaltered if we keep τ the same and take a small reduction in the residual intensity below its critical value for $\mu=0$, the photospheric intensity remaining the same. The mass supported is now slightly increased, approximately in the ratio $1/(1-\mu)$.

It appears that the least departure from an exact balance between radiation pressure and gravity at great heights causes an overwhelming concentration of the atmosphere into the lowest levels. We can make a picture of the process as follows. Consider an atmosphere, bounded internally by a plane-radiating surface, and suppose it to be fully supported by radiation pressure. Suppose now that a few more atoms are added, thereby slightly increasing the mass per unit area. We suppose $\Delta\nu$ to remain unaltered. Radiation pressure being fully occupied in supporting the existing atoms, the ones newly added will sink to the bottom. Here they will increase the absorption of radiation and cause the residual intensity in the absorption line to be slightly weaker, with the result that radiation pressure will be too small to support the atoms at high levels, or indeed anywhere, and almost the whole atmosphere will collapse on to the internal boundary surface.

The foregoing analysis contains the justification of our method of estimating τ for a Ca^+ atom from solar evidence. We assumed the calcium chromosphere to form an atmosphere fully supported by radiation pressure. If we had assumed that only $\frac{9}{10}$ of gravity was balanced by radiation pressure, we should have deduced a value for τ 10 per cent. higher. But we now see that a high-level chromosphere of the observed extent could not possibly exist if there were so much as $\frac{1}{10}$ of gravity not balanced by radiation pressure; indeed, it is questionable whether a chromosphere of the observed extent could exist if even $\frac{1}{100}$ were not so balanced. Due to the assumption $\mu=0$, then, the deduced value of τ can scarcely be in error by more than 1 per cent.

Accurate observations of the rate of decrease of the brightness of the calcium chromosphere outwards could be analysed to give the density-distribution, and a precise estimate of the upper limit of μ might be obtained. At the same time, it is clear that we must not put too much stress on our law $\rho \propto (x+x_0)^{-2}$ as giving the actual density-distribution in the high-level chromosphere. For the least alteration in F_ν below its theoretical limiting value $8\nu^2\tau mg/c$ leads to an entirely different density-distribution. We have seen that μ must be very small, but we are not entitled to assume it exactly zero. The limiting density-law is $\rho \propto (x+x_0)^{-2}$, but within a small range of μ near $\mu=0$ the distribution changes from this to a markedly exponential type. The actual density-distribution may be expected to be slightly more condensed than $\rho \propto (x+x_0)^{-2}$.

IV. *Implications of the Theory.*

15. *Relation of Chromosphere to Reversing Layer.*—The high-level calcium chromosphere appears to constitute an atmosphere fully supported by radiation pressure. The question arises, Why should this be so? As far as our analysis goes, there has appeared no reason why the sun should possess an atmosphere of Ca^+ atoms fully supported by radiation pressure rather than one partially so supported.

The answer must lie in the circumstances of formation of the chromosphere. The chromosphere is bounded below, not by a fixed radiating surface but by the upper portions of the reversing layer. In the outer layers of the star as a whole, radiation pressure will be at most some fraction of gravity which is not nearly unity, though perhaps comparable with it. I assume that this will continue to be so, in general, when, traversing the star from the interior outwards, we reach the region in which the general stream of radiation begins to show the effect of line absorption. Here, then, the mechanical equilibrium will not differ much from the ordinary gravitational type. In this portion of the star the temperature distribution will be probably very close to that given by the thermodynamic theory of radiative equilibrium for "grey" material, and if T is the effective temperature of the star (deduced from Stefan's law and the total radiation), the boundary temperature T_0 will be not far removed from the value $T \times 2^{-\frac{1}{4}} = 0{\cdot}84T$. Superimposed on this temperature gradient there may be steeper radiation-intensity gradients for particular frequencies for which there are atoms giving rise to absorption lines.* In the absence of a special intensity gradient, the residual intensity in a strong † absorption line will be that of black radiation T_0, namely, $\sigma/(e^{h\nu/kT_0} - 1)$. If a special intensity gradient becomes set up, the residual intensity will be less than this. Let us denote by F'_ν the residual intensity, in the line, which tends to be set up in virtue of the temperature gradients implied by the combined mechanical and radiative equilibrium of the reversing layer as a whole, and in virtue, in addition, of any selective intensity gradient set up subject to the same type of mechanical equilibrium. For brevity we shall say that the reversing layer tends to establish a line with a residual intensity F'_ν.

Two cases now arise. Our analysis has shown that atoms subject to an intensity $F_\nu = 8\nu^2\tau mg/c$ are just supported against gravity by radiation pressure. If $F'_\nu > 8\nu^2\tau mg/c$, the atoms near the top of the reversing layer which are selectively absorbing ν-radiation will be expelled by radiation pressure. The process will go on until an atmosphere fully supported by radiation pressure is brought into existence. It is easy to see that a state of equilibrium will be finally arrived at. For the atoms later driven off will act as a screen reducing the intensity to which the atoms first driven off are subjected, and they will themselves be subject

* A brief discussion of this has been given by the author in *Nature*, 113, 534 (1924 April 12).

† Strong in the sense of "saturated." We are referring to the limiting residual intensity, as the selective absorption coefficient for the line concerned increases indefinitely. Such coefficients, at least for "principal" lines, are known to be very large (Milne, *Proc. London Phys. Soc.*, 36, 110, 1923).

to the back-radiation from the atoms first driven off. Thus the tendency to expel atoms will go on decreasing, resulting ultimately in equilibrium. The residual intensity of ν-radiation leaving the star will then be reduced to $8\nu^2\tau mg/c$. There will be no tendency to extrude more atoms than this from the reversing layer, and hence the residual intensity will not in general sink below this value.

Should more atoms happen to be extruded, due to accidental local variations, the atmosphere fully supported by radiation pressure would collapse on to the reversing layer. We should then, at least temporarily, have an atmosphere partially supported by radiation pressure. But its equilibrium would be of exponential type, and thus practically indistinguishable from that of the reversing layer; observationally, owing to the strong condensation towards the lower levels, it would be completely indistinguishable from the reversing layer. But, in virtue of the additional pressure of the atmosphere on the reversing layer, the atoms thus precipitated would tend to be reabsorbed by the reversing layer. The normal reversing-layer gradient would tend to be set up again, and the process would be repeated.

On the other hand, if $F'_\nu < 8\nu^2\tau mg/c$, there will be no tendency to expel atoms from the reversing layer, and no chromosphere will come into existence. This state of affairs will certainly arise if

$$\frac{\sigma}{e^{h\nu/kT}-1} < \frac{8\nu^2\tau mg}{c},$$

since F'_ν is necessarily less than the photospheric intensity. But, further, it will almost certainly arise if only

$$\frac{\sigma}{e^{h\nu/kT_0}-1} < \frac{8\nu^2\tau mg}{c},$$

since, at least for a "strong" line, F'_ν cannot exceed the intensity of black radiation for the boundary temperature T_0.

For Ca^+ atoms on a star with the same value of g as the sun, it follows that in order to have a chromosphere T must not be less than 4400°, and probably, further, T_0 must not be less than 4400°, which would require $T = T_0 \times 2^{\frac{1}{4}}$ to be not less than 5230°. For a star with a value of g equal to $\frac{1}{100}$ of that for the sun, these temperatures become 2820° and 3360° respectively. If T for the sun is 6000°, the intensity for $T_0 = T \times 2^{-\frac{1}{4}} = 5040°$ would involve an intensity ratio in the H and K lines of 0·35. An observed ratio of 0·11 then necessarily implies the existence of a chromosphere, provided it can be assumed that there are no additional selective gradients in the reversing layer.*

The possibility from time to time of the collapse of the chromosphere on to the reversing layer, and its subsequent re-formation, has been shown to be a straightforward consequence of the theory. It is in agreement with the observation that the extent of the chromosphere appears to vary at different eclipses. The abrupt changes of form, position, and velocity shown by prominences are doubtless essentially the same phenomenon on a catastrophic scale, arising from sudden local

* In these calculations the value $\tau = 0{\cdot}6 \times 10^{-8}$ sec. has been used.

increases or diminutions in the corresponding ν-radiation, but it would be out of place to discuss them further here.*

16. *The Value of* $\Delta\nu$.—The observed H and K lines are very wide. On the other hand, if we could examine separately the atoms at high levels, they would be found to be emitting and absorbing radiation within a very narrow range of frequency. It has to be supposed that as we descend, due to various influences there is a dispersion of the atoms in the 1σ state or the 1π state, so that amongst them a wider range of frequencies is being absorbed and emitted. Our analysis, however, has tacitly assumed that at all depths the frequency-range for the atoms has been confined to the same interval $\Delta\nu$.

It is clear that the effect of the dispersion will be to expose the atoms, on the whole, to more radiation pressure, and therefore more will be supported; but the number supported cannot exceed that given by using in formula (45) a value for $\Delta\nu$ corresponding to the combined observed widths of the H and K lines. At high levels, however, a much smaller value of $\Delta\nu$ should be used.

Another point is that, owing to the dispersion amongst a wider range of frequencies at lower levels, the pressure at which the intensity becomes equal to the photospheric intensity will be much higher. This will in part help to bridge the gap between the pressure 10^{-12} atmos., which we have found for the base of a hypothetical atmosphere of Ca^+ atoms yielding a line 1 Å. wide, and the value 10^{-7} atmos. given by the theory of the maxima of absorption lines for the partial pressure of electrons at which the H and K lines chiefly originate.† With reference to the apparent discrepancy, it should be noticed that the electron pressure will be derived from the ionisation of many atoms other than those of calcium.

A generalisation of the theory to take account of the increasing effective width of $\Delta\nu$ with increasing depth could be sketched, but it does not seem at present profitable to do so. It seems, however, worth while to give a formal proof that the calculated value of the residual intensity applies strictly to the intensity in the centre of the observed absorption line, however wide or whatever the structure in the wings.

If at any level $n_1(\nu)d\nu$ is the number of atoms capable of absorbing radiation between ν and $\nu+d\nu$, $n_2(\nu)d\nu$ the corresponding number of atoms capable of emitting, then the total force of radiation on the gas in a thin slab is seen to be ‡

$$dxd\mathrm{S}\mathrm{B}_{1\to 2}\frac{h\nu_0}{c}\int\frac{\cos\theta d\omega}{4\pi}\int_0^\infty \mathrm{I}_\nu[n_1(\nu)-n_2(\nu)]d\nu$$

$$=dxd\mathrm{S}\frac{h\nu_0}{4c\tau\sigma_0}\int_0^\infty \mathrm{F}_\nu[n_1(\nu)-n_2(\nu)]d\nu.$$

* Our analysis has also applications to the expulsion of atoms by radiation pressure when a star suddenly brightens, *i.e.* to the phenomena of Novæ.

† Fowler and Milne, *Monthly Notices*, 84, 511, 1924. The nature of the actual "fit" of the chromosphere on to the reversing layer, given in essentials in § 15, is obscure in details.

‡ We have taken $h\nu/\sigma$ outside the sign of integration. Strictly it should be inside, but the approximation is valid, since the integrand is only sensibly different from zero near $\nu=\nu_0$.

For an atmosphere fully supported by radiation pressure we find then

$$mg = \frac{h\nu_0}{4c\tau\sigma_0} \lim_{x \to \infty} \frac{\int_0^\infty F_\nu[n_1(\nu) - n_2(\nu)]d\nu}{n_1 + n_2} \quad . \quad . \quad (66)$$

Further, for each value of ν

$$\frac{F_\nu}{2\sigma} = \lim_{x \to \infty} \frac{n_2(\nu)}{n_1(\nu) - n_2(\nu)} \quad . \quad . \quad . \quad . \quad (67)$$

which may be written

$$\frac{\sigma}{\sigma + F_\nu} = \lim \frac{n_1(\nu) - n_2(\nu)}{n_1(\nu) + n_2(\nu)} . \quad . \quad . \quad . \quad (68)$$

Assume now for $\lim [n_1(\nu) + n_2(\nu)]$ a formula of the type

$$\lim_{x \to \infty} [n_1(\nu) + n_2(\nu)] = (n_1 + n_2) A e^{-a^2(\nu - \nu_0)^2} \quad . \quad . \quad (69)$$

where

$$A \int_0^\infty e^{-a^2(\nu - \nu_0)^2} d\nu = 1.$$

By means of (68) and (69) substitute for $\lim [n_1(\nu) - n_2(\nu)]$ in (66). We find

$$\frac{4\tau cmg}{h\nu_0} = \frac{1}{s_0} = A \int_0^\infty \frac{F_\nu}{\sigma + F_\nu} e^{-a^2(\nu - \nu_0)^2} d\nu.$$

Now the actual width of the observed absorption line will be large compared with $1/a$, which is of the order of magnitude of the line-width that would be produced by the high-level atoms alone. Thus F_ν will be practically constant within the range for which $|\nu - \nu_0|$ is of the order of $1/a$. Consequently we have approximately

$$\frac{1}{s_0} = \frac{F(\nu_0)}{\sigma_0 + F(\nu_0)}$$

or

$$F(\nu_0) = \frac{\sigma_0}{s_0 - 1} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

This shows that the intensity given by the theory is the intensity in the centre of the observed absorption line.

17. *Change of Residual Intensity over the Disc.*—The fundamental equations (8) and (9) for radiative equilibrium in a particular frequency ν are identical with the standard equations of radiative equilibrium for integrated radiation (based on the thermodynamic theory), provided the "optical thickness" is taken to be

$$\int^x a(n_1 - n_2)dx,$$

and provided also that "the intensity of black radiation for the tempera-

ture at the point x," which in previous investigations * I have called B, is taken to be

$$\frac{n_2\sigma}{n_1 - n_2}.$$

Hence the results of previous investigations of the theory of radiative equilibrium can be applied as they stand to obtain better approximations than that used in §§ 7–13. We will make two applications of this theory.

I have shown that a close approximation to the law of darkening in radiative equilibrium, corresponding to a solution giving the same net flux at the boundary and in the far interior, is given by a coefficient of darkening of $\frac{3}{5}$. The intensity at the boundary is then given by

$$\begin{aligned} I_\nu &= \tfrac{5}{4}F_\nu(\tfrac{2}{5} + \tfrac{3}{5}\cos\theta), && (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi), \\ I_\nu &= 0, && (\tfrac{1}{2}\pi < \theta \leqslant \pi), \end{aligned}$$

πF_ν being the net flux. Inserting these values in (8) and integrating, we find

$$\frac{\frac{7}{8}F_\nu}{2\sigma} = \lim_{x\to\infty} \frac{n_2}{n_1 - n_2} \quad . \quad . \quad . \quad . \quad (71)$$

which is a higher approximation, to replace (18). The boundary condition (17) is unaltered.

We find then

$$F_\nu = \frac{\sigma}{s - \frac{7}{8}}, \qquad \lim \frac{n_2}{n_1 - n_2} = \frac{\frac{1}{2}\times\frac{7}{8}}{s - \frac{7}{8}} \quad . \quad . \quad (72), (73)$$

which replace (19) and (20). The resulting value of τ is practically unaltered, since s is large compared with unity.

Since on our assumptions there is strict radiative equilibrium in the radiation of frequency ν, the residual intensity in the line should obey a law of darkening over the disc corresponding to a coefficient of darkening of about $\frac{3}{5}$. Thus at the limb the residual intensity should be $\frac{2}{5}$ of the value at the centre.† The residual intensities in the H and K line at the centre and limb have been measured by Schwarzschild.‡ Inspection of his curves shows that the limb-centre ratio corresponds to a drop of almost exactly one magnitude, *i.e.* the observed ratio is about $\frac{2}{5}$, in agreement with that predicted on the assumption of a local radiative equilibrium. This is strong evidence that such a local radiative equilibrium actually exists, in the H and K radiation, and is therefore evidence as to the general correctness of the investigation of this paper.

* *Monthly Notices*, **81**, 361, 1921, etc.

† It may be recalled that for radiative equilibrium in the *integrated* radiation the law of darkening corresponds to a coefficient of $\frac{3}{5}$ for the integrated radiation but is a function of wave-length for the separate wave-lengths—the darkening is more intense in the blue than in the red. The calculated values fit very closely the observed darkening for the sun's *continuous* spectrum. But a *local* radiative equilibrium in a particular wave-length gives a coefficient of darkening of $\frac{3}{5}$ independent of wave-length, the analysis being identical with that for the integrated radiation in the former case. It would be possible for each single line in the Fraunhofer spectrum to have a coefficient of darkening of $\frac{3}{5}$.

‡ *Berlin Sitz.*, 1914, p. 1198.

Schwarzschild himself deduced from the same evidence that the lines were formed by selective gaseous scattering, to which the handing-on of quanta by successive emission, absorption, and re-emission is formally equivalent. It seems better, however, to reserve the term scattering to cases in which there is a phase-relation between the incident and scattered light, as in gaseous molecular scattering. In quantum absorption and emission there is no such phase-relation.

18. *Application to the Theory of Planetary Nebulæ.*—In a recent paper discussing the structure of the planetary nebulæ, Jeans has considered the possible effect of radiation pressure in supporting a detached shell of gas.* On certain assumptions, which he embodied in analysis, he showed that such a detached shell could exist in equilibrium. There was, however, a certain indefiniteness about his conclusions owing to uncertainties as to the value of the absorption coefficient and its dependence on temperature. In the case where the gas consists of atoms effectively capable of only two stationary states, we have seen in the present paper that such uncertainties disappear, for we separate out the effective absorption coefficient into two factors—one being the atomic absorption coefficient for an atom in the normal state, and the other being the number of atoms which are in the normal state. The latter factor is itself a function of the intensity of radiation to which the atoms are subjected, but the combined mechanical and radiative conditions of equilibrium lead to a fully determinate solution. In this case, then, we should be able to ascertain in detail the structure of a detached shell of gas, should such be possible.

Unfortunately, it can be seen very easily that a detached shell of gas, such as Ca^+ atoms, supported by radiation pressure, is impossible. For such a shell the pressure p must be zero at its internal boundary, increase to a maximum, and then decrease. Hence there must be a range in which dp/dx is positive and decreasing to zero. Now from (11),

$$\frac{dp}{dx} = mg(n_1+n_2)\left[-1+\frac{a\pi F_\nu \Delta\nu}{cmg}\cdot\frac{n_1-n_2}{n_1+n_2}\right] \quad . \quad . \quad (74)$$

and so, if there is a point where $dp/dx > 0$, at this point

$$\frac{n_1-n_2}{n_1+n_2} > \frac{cmg}{a\pi F_\nu \Delta\nu}.$$

If, further out, there is a point where $dp/dx = 0$, at this point

$$\frac{n_1-n_2}{n_1+n_2} = \frac{cmg}{a\pi F_\nu \Delta\nu},$$

and thus there must be a region in which $(n_1-n_2)/(n_1+n_2)$ decreases as we go outwards. This requires that n_2/n_1 shall increase as we go outwards. But from (14)

$$\frac{n_2}{n_1-n_2} = \frac{\frac{1}{2}(I_\nu + I'_\nu)}{\sigma},$$

* *Monthly Notices*, **83**, 481, 1923.

and I_ν and I'_ν necessarily both decrease as we go outwards. Hence n_2/n_1 must decrease outwards. It follows that a detached shell is impossible. If, indeed, there were a point at which dp/dx were positive, it would always be positive, since $(n_1-n_2)/(n_1+n_2)$ increases outwards, and the pressure would always steadily increase outwards.

Physically, the reason for this is that in an atmosphere composed of the atoms of the kind considered, radiation pressure always increases outwards. For the force due to radiation pressure is proportional to the product of the net flux and to the effective absorption coefficient; the former is constant, and the latter increases outwards because the fraction of atoms in the normal state (which alone are capable of being acted on by the pressure of radiation) increases outwards. The latter number increases outwards because the radiation intensity, which is the influence tending to transfer atoms from the normal state to the excited state, decreases outwards.

It is of interest to compare equation (74) with the corresponding equation used by Jeans. Jeans simply assumed that the beam of outward radiation decreased outwards according to the usual exponential absorption law, and used an equation which in our notation would be equivalent to *

$$\frac{dp}{dx}=(n_1+n_2)\left[-mg+\frac{a\pi I_\nu \Delta\nu}{c}e^{-\int^x a(n_1+n_2)dx}\right] \quad . \quad . \quad (75)$$

where here I_ν is constant. The exponential factor reduces the intensity as we pass outwards, and so makes it possible for dp/dx to start positive, pass through a zero, and become negative.

But Jeans' equation (75) can only be valid if radiation—the radiation from the central star—is absorbed but never emitted. It therefore implies that the shell is steadily heating up, and thus that the state is not steady. For if the state is steady, as much radiation must be being emitted as is absorbed, and there must then be a *constant* net outward flux of radiation, not one decreasing exponentially. This is true even if the assumption is made that the radiation is emitted in a different part of the spectrum. In the steady state, whatever assumption we make about the mechanism of absorption and emission, the state of equilibrium must be fixed by an equation of the form

$$\frac{dp}{dx}=n'\left[-mg+\frac{\pi}{c}\int F_\nu a_\nu d\nu\right] \quad . \quad . \quad . \quad (76)$$

where a_ν is the average atomic absorption coefficient, and

$$F=\int F_\nu d\nu=\text{const.};$$

n' is the number of atoms per unit volume. There can only be a region

* Formula (3) of the paper cited. Jeans takes account of curvature, and so replaces x by r, the distance from the centre of the star, and introduces a factor r_0^2/r^2 on the right hand side, to take account of the simultaneous inverse square decrease of gravity and of the amount of radiation per unit area. Our equations could be generalised similarly, but the presence of the inverse square factor is easily seen to be irrelevant to the present argument.

in which dp/dx is positive and then negative if there is a region in which the average atomic absorption coefficient $\bar{a}$ for all the radiation concerned,

$$\bar{a} = \int F_\nu a_\nu d\nu / \int F_\nu d\nu,$$

decreases outwards. For Ca^+ atoms absorbing and emitting the H and K radiation, it necessarily increases outwards for the reasons stated—the absorption coefficient for an atom in the normal state is constant, and the fraction of atoms in the normal state increases outwards. It seems unlikely that the average atomic absorption coefficient for any gas of constant composition could decrease outwards, for a similar reason. Where there are a variety of stationary states possible, it is of course difficult to be sure that the average atomic absorption coefficient could not decrease outwards, without making detailed calculations, and these would be very complicated, but the general conclusion seems to be that stated.

We have thus shown rigorously that a detached shell of Ca^+ atoms or similar atoms in a steady state, supported by radiation pressure, is impossible under any circumstances. And we have pointed out a grave objection to the possibility of the physical existence of any material capable of existing as a detached shell, under radiation pressure, in a steady state. It seems unlikely that a detached envelope supported by radiation pressure could exist under any circumstances unless it was absorbing more radiation than it was emitting.

If it is assumed that the envelope is rotating, the difficulty is only more acute. We can take account of rotation by replacing dp/dx by $\partial p/\partial r$ on the left-hand side of (76), and adding a term $+m\omega^2 r \sin^2\theta$ inside the square bracket on the right-hand side. Since r increases outwards, the effect of the added term is to make it still more difficult for there to be a region in which $\partial p/\partial r$ decreases outwards.

19. *Summary.** (1) *General Theory.*—Consider an atmosphere in a steady state subject to gravity and consisting of atoms capable effectively of only two stationary states, 1 and 2. Each atom alternately absorbs and emits quanta of amount $h\nu$, thereby acquiring momentum and recoil momentum respectively of amount $h\nu/c$.

Let the atmosphere be supposed to be traversed by radiation whose spectrum is that of black radiation of temperature T. The atmosphere will set up an intensity gradient in the ν-radiation, and to an observer situated outside the atmosphere the continuous spectrum of the "photospheric" radiation will appear to be crossed by a single absorption line of frequency ν. Let F_ν be the residual intensity in the absorption line, $\Delta\nu$ its breadth. The net outward flux of radiation across any level is constant and equal to $\pi F_\nu \Delta\nu$. The radiation of which the atmosphere deprives the continuous spectrum is returned by the atmosphere to the source of radiation below it, and the gas experiences, in consequence of the absorption and re-emission, a net outward force which is the pressure of radiation.

The effect of the pressure of radiation at any given level is to give

* The formulæ quoted in this summary are approximations. The exact formulæ will be found in the text.

rise to a force proportional to F_ν and to $(n_1 - n_2)/(n_1 + n_2)$, where n_1 and n_2 are the numbers of atoms per unit volume in the states 1 and 2. The ratio n_2/n_1 decreases outwards, and $(n_1 - n_2)/(n_1 + n_2)$ increases outwards. At great heights it tends to a limit, and with it the force due to radiation pressure. Let $1 - \mu$ be the limiting value of the ratio of radiation pressure to gravity at great heights. An atmosphere for which $\mu = 0$ is said to be "fully supported by radiation pressure." An atmosphere for which $\mu \neq 0$ is said to be "partially supported by radiation pressure."

For an atmosphere fully supported by radiation pressure, the residual intensity in the line is given by

$$F_\nu = 8\nu^2 \tau m g/c,$$

and the density ρ at height x is given by *

$$\rho = \frac{2\pi k T_0 \sigma \Delta\nu}{m g^2 c (x + x_0)^2},$$

where m is the mass of the atom, g is the value of gravity, τ is the "mean life of the atom in the state 2," k is Boltzmann's constant, T_0 is the temperature of the atmosphere supposed uniform, and σ is $2h\nu^3/c^2$. The constant x_0 is the height of the "equivalent homogeneous atmosphere," and taking $x = 0$ to be the level where the intensity of ν-radiation equals the photospheric value, x_0 is given by

$$x_0 = \frac{2kT_0}{mg} \cdot \frac{e^{h\nu/kT} - 1 + r}{1 - r},$$

where r is the ratio of F_ν to the "photospheric intensity" $\sigma/(e^{h\nu/kT} - 1)$.

For an atmosphere partially supported by radiation pressure,

$$F_\nu = \frac{8\nu^2 \tau m g}{c}(1 - \mu),$$

and the density ρ (except when μ is extremely small) is given by

$$\rho = \text{const.} \times e^{-\mu m g x/2kT_0},$$

the constant being determinate.

In an atmosphere partially supported by radiation pressure, the density thus falls off according to an exponential law of the usual type. In an atmosphere fully supported, it falls off extremely slowly in comparison. A change of μ from zero to even a small positive value causes an intense condensation towards the lower levels.

(2) *Applications.*—Apart from complications arising from the boundary conditions at the base, the general theory applies rigorously to Ca^+ atoms in the 1σ and 1π states yielding the H and K lines. For the sun, a change of μ from 0 to 0·01 increases the density at the base of the atmosphere by a factor of 6 and decreases that at a height of 14,000 km. by a factor of a million. For $\mu = 0$ the density diminishes

* The formulæ are stated for singly ionised atoms accompanied by an equal number of free electrons. For neutral atoms, replace $2kT_0$ by kT_0.

by a factor of $\frac{1}{2\cdot3}$ in 14,000 km.; for $\mu = 0\cdot01$ the factor is 10^{-8}; for $\mu = 0\cdot1$ it is 10^{-80}. The observed extent of the Ca^+ chromosphere is incompatible with any theoretical exponential density law, and in view of these calculations we conclude that the high-level calcium chromosphere must practically constitute an atmosphere fully supported by radiation pressure; the value of μ can scarcely be so large as 0·01.

The observed value of F_ν for the H and K lines then provides a determination of the constant τ for the 1π state of the Ca^+ atom; it is found to be $\tau = 0\cdot6 \times 10^{-8}$ sec., the same as found by a slightly different method in a previous paper. The constant x_0 is about 3660 km. If μ is not exactly zero, the density will decrease somewhat more rapidly than is given by $\rho \propto (x + x_0)^{-2}$. For $\Delta\nu$ corresponding to $\Delta\lambda = 1$ Å., the pressure at the base is about $\frac{1}{2} \times 10^{-12}$ atmos., the density is about 3×10^{-17} gram. cm.$^{-3}$, and the number of atoms per unit volume is 4×10^5; the total number of atoms per sq. cm. column is about $1\cdot5 \times 10^{14}$.

An atmosphere partially supported by radiation pressure is so strongly condensed towards its base that it can hardly be distinguished from the reversing layer itself, either theoretically or observationally. We thus see how an atmosphere fully supported by radiation pressure comes into existence. The reversing layer by itself tends to establish an absorption line of a certain residual intensity, say F'_ν. If $F'_\nu < 8\nu^2\tau mg/c$, no "chromosphere" will exist; if $F'_\nu > 8\nu^2\tau mg/c$, atoms will be expelled from the reversing layer by radiation pressure until a layer of atoms supported by radiation pressure is formed sufficient to reduce F'_ν to the value $8\nu^2\tau mg/c$. These atoms will constitute an atmosphere fully supported by radiation pressure, and will be observed as a chromosphere.

A star with the same value of g as the sun, of effective temperature less than 4400°, is incapable of supporting a Ca^+ chromosphere under any circumstances. Further, if the reversing layer tends to set up a residual intensity not greater than that of black radiation for the Schwarzschild boundary temperature (as is probable for a "saturated" line), the effective temperature must be not less than $4400° \times 2^{\frac{1}{4}} = 5230°$. For a giant star with a value of g equal to $\frac{1}{100}$ of that for the sun, these temperatures become 2820° and 3360°.

Some of the conclusions of this paper are in conflict with those recently obtained by Stewart, working on the basis of the classical theory of radiation. They do not involve the difficulties encountered by Stewart.

The observed change of residual intensity in the H and K lines from the centre of the sun to the limb is in agreement with the theory of a local radiative equilibrium in the H–K radiation, as assumed in this paper.

A detached shell of Ca^+ atoms supported by radiation pressure, of the kind contemplated by Jeans in his theory of planetary nebulæ, cannot exist in equilibrium. It is improbable that a detached shell of gas supported by radiation pressure can ever exist in equilibrium unless it absorbs more radiation than it emits, *i.e.* unless its thermal state is not steady.

The Condensations in the Spiral Nebulæ. By J. H. Reynolds.

The condensations in the spiral nebulæ, first observed visually by Lord Rosse and interpreted by him as being the precursor of resolution of all the nebulæ into stars, have been shown by photography to be almost invariably present in the developed spirals. They evidently have an important bearing on the history of the development of these formations, as they are few in number in the spirals which consist of amorphous nebulosity with large bright nuclei, and increase rapidly in number with increasing extension of the spiral arms and a resulting loss of matter in the nuclei. Jeans has dealt specifically with these features in his *Problems of Cosmogony* (p. 216 *et seq.*). He first considers the conditions under which a filament can continue to exist without scattering into space, and shows that for an isothermal filament the critical value of the line density, $T, = \frac{2K}{\gamma} = \frac{2C^2}{3\gamma}$, C being the probable molecular velocity and γ the gravitation constant. For line densities greater than this the filament may become transversely as well as longitudinally unstable. Basing his conclusions on the probable rotation period of 85,000 years, found by Van Maanen for the condensed spiral M 101, Jeans finds that ω (the mean angular velocity) $= 2{\cdot}35 \times 10^{-12}$ and if C be taken as 1·6 km./sec., the distance between two adjacent condensations $l = 7 \times 10^{11}$ km. (about $\frac{1}{45}$ parsec).* Jeans therefore provides us with an important theoretical method for determining the probable distances of the spirals, and the large amount of observational material now in existence in the form of photographs makes it possible to undertake a general survey of the question of its applicability.

An examination of the photographs shows at once that the spirals vary greatly in the amount of matter contained both in the nuclei and the spiral arms. This variation is not confined to either of these features, but applies apparently to the whole nebula. We may take as comparable examples M 101 (N.G.C. 5457) and M 33 (N.G.C. 598). Both these spirals are of large angular diameter, contain well-marked condensations, and would be classed together as "much condensed," the nuclei being comparatively faint and small compared with the spirals consisting principally of amorphous nebulosity. In the case of M 101 the spiral arms are narrow and filamentous, some isolated condensations being found in extension of the irregularly developed arms as far away as 14′ from the nucleus to the north, although to the south the spiral has a definite boundary at half this distance. In places the condensations are far apart, and vary much in apparent size and brightness. In marked contrast to this spiral, M 33 contains an enormous number of small, stellar condensations spread over the whole breadth and length of the arms, which practically form a continuum, as they cover the whole area without intervening dark spaces such as are found in

* The argument is dependent on the value of ω, so the theoretical distance apart of the condensations would have to be modified if the rotation periods for the spirals measured by Van Maanen should prove to be longer or shorter than the periods given by him. Another twenty years will see these measures definitely confirmed or otherwise.

M 101. The regions around the nebulous area also contain many small star-like objects which obviously originated in the nebula. In the case of M 33 there are, therefore, no isolated filaments, and the formation of the condensations has taken place over large areas. This agrees with a general density considerably greater than the minimum line density postulated by Jeans' hypothesis, and it appears that we must assume great variations in density between different objects. We may therefore divide the spirals of a similar degree of condensation into two classes, the massive and the filamentous, and it will be found that this distinction applies to all degrees of condensation. The question arises whether the mechanism of the formation of condensations in the two classes is the same. There is certainly a great difference in the appearance of the condensations: in the case of the massive class, they are small, well defined, and almost stellar, with a tendency to close grouping; whereas in the filamentous class some of them are large, irregular, and of varying degrees of brightness, while faint and small condensations are also present. In general the large bright condensations are found widely separated in the outer regions of spirals, while the smaller and fainter are crowded together nearer the nucleus. One of the best examples of the filamentous class is N.G.C. 628. The condensations are well marked, the plane of the spiral being approximately at right angles with the line of sight, and this has been selected for a detailed measurement of the condensations. Like all condensed spirals it has developed considerable irregularities of form with subsidiary bifurcations of the arms at intervals. Only one of the two antipodal arms is regularly developed in the spiral form. The other spreads out irregularly after one revolution of the spiral. In the table (p. 144) θ = the position-angle with respect to the nucleus, d_1 = the angular distance from the nucleus, d_2 = the angular distance between the particular condensation and the preceding one measured outwards along the spiral filament, m = the brightness of the condensation on an arbitrary scale of 10. The condensations near the nucleus are indefinite in themselves and hardly measurable as separate objects, so only those which are well marked and at some distance from the nuclear point have been measured.

Small faint condensations also occur between the primary condensations given in the table.

It is apparent that there are two types of condensation present in this spiral, one small and faint, the other large, bright, and irregular. Star discs also appear in the outer regions along the spiral arms, but it is doubtful whether these belong to the nebula or are part of the general stellar field. The spiral M 101 (N.G.C. 5457) is another example of the same type, but in this case, instead of the large primary condensations in N.G.C. 628, appear patchy masses of irregular form, interspersed with faint, well-defined stellar condensations. The distances between the condensations vary greatly, from 5″ (of which there are a considerable number) up to several times this amount. Many star discs of about 16th magnitude appear along the arms of the spiral, and it seems probable that a good proportion of the stars in the field have originated in the nebula. It is of such an irregular form that detailed measures would lead to no results which could be consistently interpreted. A smaller

TABLE I.

Measurements of the Condensations in N.G.C. 628.

No.	θ.	d_1.	d_2.	m.	Character.
	°	′ ″	′ ″		
1	0	1 33	...	3	Small round.
2	7	1 43	0 21	3	,, ,,
3	20	1 38	0 21	2	,, ,,
4	32	2 4	0 26	3	Irregular.
5	46	2 9	0 36	3	Small round.
6	55	1 53	...	8	Probably star about 15th mag.
7	64	1 53	0 41	4	Diffuse.
8	82	1 54	0 36	7	Double.
9	107	2 0	0 51	6	Multiple and irregular.
10	133	2 14	0 56	7	Well defined.
11	150	2 28	0 52	6	Irregular, diffuse.
12	160	2 46	0 31	8	Large, irregular, diffuse.
	Several small condensations, mag. 1 and 2, irregularly disposed, appear here.				
13	193	3 27	2 0	8	Large, diffuse.
14	231	3 52	2 30	10	Star about 14th mag.
15	260	4 23	2 14	9	,, ,,
16	306	4 22	3 37	5	Well defined.
17	333	4 17	2 5	4	,, ,,

but very beautiful example of the filamentous type of condensed spirals is N.G.C. 1232. It consists of a small bright nucleus with irregularly branched filamentous arms along which the condensations appear usually in pairs. Its diameter is 5′, and the arms being narrow and continuous represent Jeans' theoretical filament perhaps better than any other example. The inner condensations are spaced at intervals of 2″ and the outer about three times this amount. A great contrast to the filamentous type of condensed spirals is afforded by the massive type, of which M 33 is the most striking example, both for its angular size and for the enormous number of condensations it contains. There is a very marked tendency to grouping, and the outer regions are sometimes free from nebulosity, the condensations appearing as small round discs, differing little if at all from star discs of about 17th and 18th magnitude. The Andromeda Nebula (M 31), although not of the condensed type over the greater part of its area, has a similar kind of condensation in the outer regions of the spiral both in appearance and in their small average spacing. The following table includes spirals of various types, with reference to the condensations found in each. A distinction is made between the angular spacing of the condensations both in the inner and outer regions, as it is almost invariably found that the spacing increases outwards. In some spirals the spacing is so irregular that it is useless attempting to give a definite average interval. In others, on the other hand, the spacing is remarkably regular.

TABLE II.

N.G.C.	λ.	β.	Δ.	Type.	Condensations. Inner.	Condensations. Outer.	Description.
224	87°	−22°	130′ × 45′	Massive	...	3″–10″	Stellar condensations only in the outer region.
253	83	−86	18 × 5	,,	....	? 35″	Patchy and indefinite.
300	265	−77	20 × 10	,,			
578	158	−80	4 × 3	Fil.	...	30″	
598	101	−31	80 × 60	Massive	...	5″–25″	Great numbers of stellar condensations crowded together in places.
613	200	−79	5 × 3·5	Fil.	20″	Irr.	Well marked.
628	105	−46	9 × 8	,,	21″	1′–3′	Large and small isolated condensations.
891	108	−18	10 × 13	Edge on	...	40″	Few only visible.
1232	178	−58	5 × 5	Fil.	5″	...	In pairs, well marked.
1403	118	+27	10 × 65	Massive	12″	24″	Large and small condensations.
2835	220	+21	6 × 5	Fil.	...	40″	
2841	135	+45	6·5 × 2·2	Massive	...	28″	Condensed in outer regions only.
2903	177	+43	12 × 5	Multi fil.	...	20″–40″	Indefinite.
3031	107	+39	16 × 10	Massive	...	...	Outer regions only condensed.
3198	137	+55	9 × 3	Fil.	15″	25″	Well marked.
3027	209	+66	8 × 2·5	Massive	...	12″	Little condensed, irregular.
3726	118	+64	6 × 3·5	Fil.	10″	25″	Well marked.
4254	240	+76	5 × 5	,,	12″	25″	,, ,,
4258	143	+67	20 × 6	Irregular	...	45″	Little condensed.
4303	252	+68	6 × 6	,,	...	25″–40″	Outer ring condensed.
4321	238	+78	5 × 5	Fil.	11″	25″–40″	Well marked.
4485	101	+75	4 × 1·8	Very irreg.	...	10″–20″	Not spiral in form.
4536	262	+66	7 × 1·8	S-shaped	14″	40″	
4567	262	+74	2 × 1	Fil.	6″	12″	
4568	260	+75	3 × 1	,,	...	12″	
4725	50	+82	5 × 4	Barred ring	...	26″–40″	Ring only condensed.
4736	87	+75	5 × 3·5	Multi-spiral	14″	...	Inner portion only condensed.
5055	75	+74	8 × 3	Massive multi-spiral	...	14″–30″	Outer regions condensed.
5194	69	+68	12 × 6	Fil.			
5236	282	+32	10 × 8	,,	15″	Irr.	Well marked.

TABLE II.—*continued.*

N.G.C.	λ.	β.	Δ.	Type.	Condensations. Inner.	Condensations. Outer.	Description.
5247	284°	+45°	5′ × 5′	Fil.	15″	45″	
5383	48	+69	2·25 × 2·25	S-shaped	...	5″	
5457	168	+59	16 × 16	Fil.	10″–20″	15″–40″	Spacing irregular.
5859	357	+57	2·6 × ·8	,,	...	14″	Outer whorl only co densed.
6070	336	+36	3 × 1·6	,,	...	12″	
6946	61	+12	8 × 8	Inter-mediate	...	14″–20″	In rich star-field.
7217	54	−19	2·5 × 2	Ring	...	20″	
7331	60	−20	9·5 × 2	Fil.	...	34″	Outer region only co densed.

It will be seen from the table that while spacing of the condensations in the spirals of large angular diameter is on the whole greater than in the smaller examples, the two greatest of all, M 13 and M 33, both contain faint stellar condensations with a spacing small compared with their large diameter. It may well be that exposures with reflectors of great aperture may modify the results obtained, but the majority of the photographs examined were of excellent definition. Most of the condensations show spacing of 15″ and upwards, and on Jeans' hypothesis we must regard many of the spirals of moderately large angular diameter (10′ to 20′) as much closer than the giants M 31 (130′ × 45′) and M 33 (80′ × 60′), and although the angular diameter is no doubt to some extent an indication of distance, there seems to be very distinct evidence that there is a great disparity both in extent and mass between individuals. If we take the spacing of the condensations as a criterion, N.G.C. 1232 and M 31 are approximately the same distance away, although the former is only 5′ in diameter against 130′ for the latter. There is also the distinction that N.G.C. 1232 is composed of fine filaments while M 31 is an outstanding example of the massive type, and if the plane were at right angles to the line of sight would cover an area of sky over 600 times greater than N.G.C. 1232. The concentration of the spirals of great angular diameter in the celestial hemisphere containing only 5 globular clusters out of 81 is a remarkable fact, and it is difficult to avoid the conclusion that the spiral and spheroidal nebulæ are in some way the material from which the star clusters have ultimately developed, although there is a complete lack of transitional examples between the two forms.

An interesting question with regard to these nebular condensations is that of their evolution into stars of ascending temperature. We should expect first of all that the condensations in the filament would gradually concentrate into M-type giants of short effective wave-length. As the outer regions of the spirals are usually the first to form condensations, these regions would be developed first and the resulting stars would be at a more advanced stage than the condensations nearer the nucleus. As

this would correspond to an increase in temperature and therefore to a shorter effective wave-length, there should be an increase in photographic magnitude of the stellar condensations from the nuclear to the outer regions of the spiral. This certainly seems to be the case in a great many instances. Perhaps the most striking example is the much condensed spiral M 33. In the outskirts of this spiral appears the bright-line condensation N.G.C. 591, which was used by Slipher for determining the radial velocity of the whole nebula (−263 km./sec.). This type of nebulosity is associated only with stars of B type and a probable temperature of 15,000° C. or more, so we are on safe ground in inferring a considerable advance in development in these outer regions, and the star discs here are generally of greater photographic magnitude than those in the nuclear regions. It might be expected that photographs with panchromatic plates and various screens transmitting only a portion of the spectrum would show differences in effective wave-lengths in the stellar condensations, but this method is complicated by the fact that the contrast factor for panchromatic plates is greater for the red than the blue region of the spectrum (*vide* Ross, *Ap. J.*, **52**, 80). The results obtained by Seares some years ago, however, on the condensed spiral M 99 (N.G.C. 4254) were by means of a yellow screen and an isochromatic plate compared with an unscreened ordinary plate. As far as star discs and the nucleus of the nebula are concerned, the exposures gave similar results, but the condensations, quite apparent in the blue image, were almost entirely absent in the yellow image. The contrast factor for the isochromatic plate does not vary much with the wave-length, so that the effect is not likely to be a photographic one; and the conclusion drawn by Seares, that the condensations are deficient in the longer wave-lengths and therefore have a large negative colour index, seems justified. As the point is an important one, a repetition of this piece of work under more rigorous conditions would be desirable. The ordinarily accepted stellar progression from M type to F type would not agree with this large negative colour index, which indicates rather the spectrum of gaseous nebulæ and stars of types B and O. The case of the bright-line condensation in M 33 has already been referred to, and there is a probability that many of the outer condensations of the spirals are gaseous and connected with high-temperature stars.

The general conclusion reached by the present paper on the question of "island universes" is that while the great massive spirals M 31 and M 33 are possibly comparable in size and mass with the galactic system, the great majority of spirals are intrinsically small in comparison, being filamentous in structure, with comparatively few condensations. M 99, for example, which is much condensed even close to the nucleus, contains no more than 100 stellar condensations.

On Large Line-Displacements in the Spectra of Germanium and Chlorine under Different Conditions of Temperature. By Joseph Lunt, D.Sc. (Plates 4, 5.)

(*Communicated by H.M. Astronomer at the Cape.*)

During an investigation of the spectrum of germanium, which has been dealt with in a separate paper,* a very large difference in the wave-lengths of two of the red lines was obtained when the values from the metallic spark in air at atmospheric pressure were compared with those obtained from a vacuum-tube of germanium chloride.

It was also found that very large shifts of lines both of germanium and chlorine could be obtained in consecutive exposures from a vacuum-tube of germanium chloride under different conditions of excitation which gave rise to great changes of temperature and brightness.

The provisional wave-lengths of the two red lines of germanium are :—

Low Temperature.	High Temperature.	$\Delta\lambda$.
6336·4	6337·9	1·5
6484·4	6486·0	1·6

So recently as 1907 the question of the invariability of the wave-lengths of spectrum lines was much discussed, and the eminent authority on spectroscopy, Professor Kayser, in a paper "On the Constancy of Wave-lengths of Spectral Lines,"† wrote as follows :—"At the meeting of the Union in Paris in May (1907), Professor Ames was able to communicate the fact that Dr. Pfund had made experiments in his laboratory with the interferometer which proved that the wave-lengths are precisely the same, regardless of whether the spectrum was produced in the spark or in the arc, at atmospheric pressure or in a vacuum, of pure metals or of an alloy or salts. This was true without exception for all the elements investigated. Professor Fabry declared that his experiments had yielded precisely the same result. Inasmuch as the most precise method which we have was employed here, we must regard these experiments as decisive, and consider that the question of the constancy of wave-lengths is finally settled." Unfortunately, in spite of the above most definite pronouncement, later work has shown that variations in wave-length do occur, and that, in order to obtain an accurate set of standard lines of iron for use in spectroscopy, most minute precautions have to be taken to use a source of light produced under uniform conditions, and even then to reject lines which are liable to variations.

Even now it would appear that finality has not been reached in the method to be adopted to obtain constant wave-lengths of iron lines as standards. In the recent very important paper by Meggers,‡ he says: "From what has been said it may be concluded that internal evidence from the structure of the spectrum (of iron) shows that even

* *M.N.*, 85, 38.
† *Astroph. Journ.*, 26, 191 (1907).
‡ *Ibid.*, 60, 74, 1924 July.

the best values of wave-lengths now available from the iron arc in air do not form a perfect system of standards. . . . The iron arc at atmospheric pressure gives values among which different groups are affected differently by pressure and by pole-effects. . . . It is believed that the iron arc *in vacuo* will furnish a more perfect and useful system of standards than the one now in use," which is quite in accord with the experience of Fabry and Buisson,* who say: "Si, au lieu de partir de l'arc dans l'air, on compare directment le spectre solaire avec celui de l'arc dans le vide, l'élargissement dissymétrique de la raie d'émission est supprimé, ou du moins très atténué; les anomalies résultant des dissymétries disparaissent et toutes les différences sont de même signe."

Whilst this variability of wave-length adds greatly to the complications of spectroscopic work, it may yield very valuable information of an astrophysical character when the variations of wave-length have been more fully investigated and their causes determined.

The present paper describes some experimental work on the spectrum of chlorine which shows very large variations of wave-length under different conditions of temperature in vacuum-tubes.

The temperature attained by the gas in a vacuum-tube depends on at least a dozen factors. The conductivity of a capillary tube depends on (1) the length, (2) the internal diameter, (3) the nature of its gaseous contents, (4) the temperature and pressure of the gas.

Working with an induction coil and a mercury-jet break, the temperature depends upon (5) the voltage on the primary, (6) speed of motor, (7) duration of the contacts, (8) number of breaks per revolution, (9) the amount of capacity, (10) the self-induction, (11) length of spark gap introduced into the secondary circuit, and (12) the size and efficiency of coil.

For a given tube the temperature and luminosity are, in practice, varied by altering conditions 5–12 until a suitable exposure time is arrived at. In the present work the lowest temperatures were obtained by using a 6-inch spark coil with no capacity, self-induction, or spark gap in the secondary circuit. Intermediate temperatures were obtained by the use of a single small plate condenser of 0·0025 μF capacity, self-induction 183 μH, and no gap. Higher temperatures were obtained by using two plates of capacity 0·005 μF, no self-induction, and a spark gap.

Sometimes an 18-inch spark coil is used to obtain higher temperatures, but glass tubes usually become corroded through the decomposition of the silica of the glass and the capillary cracks in many places owing to the high internal temperature and unequal expansion.

The vacuum-tubes used were of two kinds: (1) the ordinary Geissler form with a capillary about 8 cm. long and about 1 mm. internal diameter; (2) end-on tubes with a shorter capillary 4 cm. long and the same internal diameter.

It was found that a condensed discharge would not pass through the ordinary long Geissler tube when filled either with germanium tetrachloride or tin tetrachloride to a pressure of 12 mm., and that the uncondensed discharge would pass and was sufficient to dissociate these gases and give the spectra of their constituents, as was also found

* *Astroph. Journ.*, **31**, 113, 1910 March.

by Hagenbach and Konen * in the case of tin tetrachloride. The germanium-tetrachloride tubes when sealed off from a reservoir of gas soon deposited the metal in the cooler bulb, but fortunately left the hotter capillary clear. When this process is completed a great diminution of luminosity and a change of colour is seen, and it is then possible to pass the condensed discharge and so obtain the spectrum of the residual chlorine with but a slight admixture of the germanium spectrum. The germanium spectrum can be regained by gently heating the bulb and regenerating the chloride of the metal.

Tin tetrachloride does not lose its metal so rapidly, and can be used for a much longer time. From the wave-lengths of the unstable chlorine lines it seems that the temperature in the capillary must be higher than in the case of a similar tube of germanium chloride under the same electrical conditions of working, and the recombination of its dissociated elements must take place more easily.

The tubes were filled after a thorough exhaustion and elimination of as much occluded hydrogen as possible, by introducing the liquid chlorides at the foot of a barometric column of mercury at the head of which was a reservoir communicating with the vacuum-tubes. In the case of tin tetrachloride the vapour pressure at laboratory temperature was found to be about 12 mm., and so the saturated vapour could be used, but in the case of germanium chloride the first introduction of the liquid gave a pressure which had subsequently to be reduced by pumping until it reached 12 mm.

The interior of the pump-and-tube system must be perfectly dry, or decomposition of the chloride takes place and the glass loses its clearness from the deposition of hydrated oxide. If air is subsequently introduced it should be thoroughly dried to avoid this decomposition.

A great many chlorine lines are subject to very large shifts amounting to 0·8 Å. in the H_γ group, and these shifts are regarded as solely due to the changes in temperature of the gas. There can be no question of the shifts being of instrumental origin, as lines with no pronounced shift occur side by side with the shifted lines which I propose to call unstable lines. They are so numerous in the spectrum of chlorine that only two typical groups are dealt with in this paper, viz. the H_γ group and the ultra-violet group, leaving the others to be dealt with in a separate paper on the chlorine spectrum, which will extend our knowledge to the infra-red.

Table I. gives the measures of the lines in the first group, divided into two series. In the first series the unstable lines are shown. In the second are given lines which show no obvious displacement on mere inspection, but even in these the definite progression in the wave-lengths on passing from low- to high-temperature conditions indicates that the wave-lengths are really variable, but to a much smaller degree than is found in the first series.

Table II. gives the measures for the ultra-violet group. The smaller displacements found for this group only indicate that the range of temperature has not been pushed so far as in the preceding group, but the results are sufficient to show that large variations of wave-length occur.

* *Atlas of Emission Spectra*, p. 49.

In tabulating the lines in the spectrum of chlorine it is clearly advisable to work at the lowest possible temperature, not only on account of the diffuseness of the lines at higher temperatures, but to obtain a more fundamental value for the wave-lengths.

Large discrepancies between the wave-lengths of chlorine lines given by Eder and Valenta and Exner and Haschek are seen to be explained by the different temperatures employed by them, and the differences in wave-length recorded by Eder and Valenta in chlorine at different pressures are also probably due to temperature differences.

The unstable lines become very much more diffuse at higher temperatures, but the diffuseness does not consist merely in a more or less pronounced shading to the red with the strongest part of the line in its original low-temperature position, but the centre of gravity of the lines is shifted to the red and both edges are diffuse.

Moreover, the shifts are progressive, and it seems possible to place the lines in a definite temperature scale from the measures of their wave-length alone. These results open up a wide and important field of research, and may explain many of the discordances observed in the radial velocities of stars given by different lines in the spectrum which cannot be explained by blending. They indicate too that the wave-lengths of individual lines, apart from differences in relative intensities, may give valuable information as to stellar temperatures. Plates 4 and 5 show enlargements of the spectra dealt with. The spectrograph employed is described in a previous paper.*

There is abundant experimental evidence to show that many spectral lines are subject to a temperature-shift, but this effect of temperature-change seems not to have been stressed so much as its importance deserves. The tendency has been to ascribe variations of wave-length in solar and stellar spectra to pressure shifts—which have given values for the pressure in the reversing layer of the sun which have had to be abandoned—or to Doppler effects. As regards the latter, the question arises as to whether different velocities are found for lines affected and unaffected by temperature-shift.

In the laboratory King † has obtained abundant evidence in the spectra of titanium and iron to show that the wave-lengths of lines given by the cooler vapours producing absorption are shorter than those produced at higher temperatures, and that this effect is not general but confined to certain groups of lines, other groups being unaffected, just as has been found in the case of chlorine, although the amount of shift is very much smaller. Some of the iron lines showing this temperature-shift have been produced as emission lines in the flame spectrum by Hemsalech and de Watteville,‡ and it would be of interest to trace the changes in wave-length from the lowest to the highest temperatures obtainable in the laboratory with sufficient dispersion to bring out the results clearly.

King § suggests *density of high-speed electrons* in preference to vapour density or density of ions suggested by Royds, but, as Saha ‖

* *M.N.*, **85**, 38.
† *Astroph. Journ.*, **41**, 373, plates vii. and ix.
‡ *Comptes Rendus*, **146**, 963.
§ *Astroph. Journ.*, **41**, 393.
‖ *Phil. Mag.*, **41**, 278.

points out, "The spark produces mechanically *the very same conditions* [italics mine] which can be realised at very high temperatures," and here seems a danger of replacing the word temperature by other words which mask its significance.

Other lines showing a temperature-shift as revealed by the dissymmetry of their reversals are the D lines of sodium * and the H and K lines of calcium; † in these cases the shift at higher temperature appears to be to the violet. Kayser's *Handbook*, vol. ii., plate iii., shows the calcium lines (Ca^+) 3159, 3179, 3181 with the absorption line on the violet side of the emission line. To these lines I may add the strong potassium pair 4044, 4047, which I have found to stand out amongst accordant lines in the spectrum of lead-pencil as of longer wave-length than the solar lines.

Exner and Haschek record this pair as reversed, with the red component stronger. King † states that "no reversed iron lines with the violet side strongest have been found," but an important group in the green-yellow "were found to give a decided displacement towards the violet," so we may have three groups: (1) unaffected, (2) shifted to red, (3) shifted to violet.

The atom is such a complicated structure that its different component parts or disintegration products may respond differently to enhanced temperatures.

As the poles of the arc show enhanced lines, a proof of a higher temperature than exists in the centre of the arc, the shifts recorded as "pole-effects" may really be due to a difference of temperature.

In stars, St. John and Adams ‡ conclude that "the relative displacement of arc and enhanced lines . . . may be correlated with temperature," but ascribe the shifts to Doppler effects.

In the spectrum of ρ Cassiopeiæ, Adams, Joy and Sanford § find "the enhanced lines are very strong, and appear to be displaced to the red with respect to the arc lines."

The work of Albrecht ‖ showing the progressive variation in the wave-lengths of stellar lines on passing from type to type of stars is well known, and he proposed to determine quantitatively the type ¶ of a star from the values obtained for the wave-lengths of the variable lines. In stellar spectra the influence of variation in strength of the components of a blend is difficult to eliminate. Leaving blending out of account, the variations of wave-length on passing from types F to Mb only exceed 0·2 Å. in 7 out of 40 lines given in his table II., but the range is limited to the region from λ 4246·9 to λ 4473·0.

In a paper on the air lines in spark spectra, Merrill ** records some observations which have a distinct bearing on the question of changes in wave-length of lines of nitrogen, argon, and hydrogen. Merrill bases

* *The Observatory*, **47**, 11 (Evershed), 1924 January. See also Kayser's *Handbook* vol. ii., plate ii., No. 4.

† *Astroph. Journ.*, **41**, 377.

‡ *Astroph. Journ.*, **60**, 43, Abstract (2).

§ *Pub. Ast. Soc. Pac.*, **36**, 138 (footnote).

‖ *Astroph. Journ.*, **33**, 130 (1911).

¶ We may substitute "temperature" for "type."

** *Astroph. Journ.*, **51**, 238.

his conclusion as regards nitrogen on a comparison of his values derived from spark spectra in air at atmospheric pressure with Porlezza's values for vacuum-tube spectra. Porlezza * used a helium comparison spectrum and Eversheim's wave-lengths; the available comparison lines are too few to give a satisfactory standardisation, and long stretches of spectra are without reference lines.

Moreover, Porlezza's nitrogen spectrum in the region 3995 to 4641, where it is possible to compare his values with my own (unpublished) and with Fowler's, shows that whilst the latter two give a mean difference for 30 lines of only 0·01 Å., Porlezza's differ from Fowler's 0·28 Å. in the sense

Fowler–Lunt . . .	−0·01 Å.
Fowler–Porlezza . . .	−0·28 Å.

In the region of the yellow lines, Merrill's observations agree fairly well with Exner and Haschek's values, and both agree with my own values both at low and high temperatures from vacuum-tubes, but again Porlezza's wave-lengths are discordant, as the following figures show :—

Merrill. Spark in Air.	Exner and Haschek. Spark in Air and in N.	Lunt. Vacuum-Tube, N.	Mean 1, 2, 3.	Porlezza. Vacuum-Tube, N.	(4)–(5).	(3)–(5).
(1). I.A.	(2). I.A.	(3). I.A.	(4). I.A.	(5). I.A.	I.A.	I.A.
5927·83	27·83	27·87	27·84	27·57	−·27	−·30
31·78	31·83	31·80	31·80	31·50	−·30	−·30
	40·48	40·31	40·40			
41·54	41·68	41·61	41·61	41·49	−·12	−·12
52·33	52·38	52·47	52·39	52·20	−·19	−·27

It is therefore unsafe to assert that there is a large difference between the wave-lengths of nitrogen lines obtained from vacuum-tubes and from the spark in air without further evidence. It is nevertheless desirable to investigate the spectrum in vacuum tubes at low and high temperatures in order to ascertain whether anything analogous to the cases of the lines of germanium and chlorine takes place.

My sincere thanks are gratefully tendered to Mr. J. W. Jackson, who has rendered invaluable assistance throughout the investigation; to Professor J. Smeath Thomas of Cape Town, who kindly prepared and presented the liquid chlorides of germanium and tin; and to Mr. H. Spencer Jones, H.M. Astronomer, for his kindly interest and encouragement in the work.

Royal Observatory, Cape of Good Hope:
1924 *October* 28.

* *Atti R. Accademia dei Lincei*, **20**, 585 (1911).

TABLE I.

Temperature-shift of Chlorine Lines, H_γ Region.

Low Temperature.								High Temperature.							
Lunt.		Eder and Valenta.				Lunt.		Lunt.							
A 259(40) $GeCl_4$ End-on Tube.		(a) 10–20 mm.		(b) 30–40 mm.		A 278(20) Cl in $GeCl_4$ Capillary.		A 259(30) $GeCl_4$ End-on Tube.		A 284(30) $SnCl_4$ Capillary.		A 278(50) Cl in $CeCl_4$ Capillary.		A 278(40) Cl in $GeCl_4$ Capillary.	Range highest to lowest.
(1).	(2)-(1).	(2).	(3)-(2).	(3).	(4)-(3).	(4).	(5)-(4).	(5).	(6)-(5).	(6).	(7)-(6).	(7).	(8)-(7).	(8).	
4225·01				25·14	+ ·17	25·31	+ ·02	25·33	+ ·19	25·52	+ ·22	25·74	+ ·10	25·84	+ ·83
34·06	+ ·08	34·14	+ ·06	34·20	+ ·10	34·30	+ ·23	34·53	+ ·13	34·66	+ ·06	34·72	+ ·16	34·88	+ ·82
35·58	+ ·03	35·61	+ ·07	35·68	+ ·09	35·77	+ ·23	36·00	+ ·18	36·18	+ ·10	36·28	− ·13	36·15	+ ·70
41·35	+ ·09	41·44	+ ·03	41·47	+ ·13	41·60	+ ·23	41·83	+ ·18	42·01	+ ·09	42·10	+ ·12	42·22	+ ·87
53·47	+ ·06	53·53	+ ·11	53·64	+ ·13	53·77	+ ·24	54·01	+ ·19	54·20	+ ·08	54·28	+ ·08	54·36	+ ·89
61·31	+ ·04	61·35	+ ·07	61·42	·00	61·42	+ ·27	61·69	+ ·22	61·91	+ ·08	61·99	+ ·13	62·12	+ ·81
70·72	+ ·01	70·73	+ ·13	70·86	+ ·04	70·90	+ ·23	71·13	+ ·21	71·34	+ ·06	71·40	+ ·19	71·59	+ ·87
76·60	+ ·03	76·63	+ ·09	76·72	+ ·12	76·84	+ ·26	77·10	+ ·14	77·24	+ ·11	77·35	+ ·10	77·45	+ ·85
Mean diffs.	+ ·05		+ ·08		+ ·09		+ ·22		+ ·18		+ ·10		+ ·09		+ ·83
Lines with no Pronounced Temperature-shifts, same Negatives.															
4132·63	+ ·05	32·68	+ ·04	32·72	− ·08	32·64	+ ·03	32·67	+ ·01	32·68	+ ·06	32·74	+ ·01	32·75	+ ·12
57·93	+ ·09	58·02	− ·02	58·00				58·07	− ·01	58·06				58·04	+ ·14
4259·62	+ ·01	59·63	+ ·01	59·64	+ ·07	59·71	− ·02	59·69	+ ·02	59·71	+ ·10	59·81	− ·09	59·72	+ ·19
91·85	+ ·01	91·86	+ ·02	91·88	+ ·02	91·90	+ ·03	91·93	+ ·01	91·94	− ·01	91·93	+ ·02	91·95	+ ·10
4304·18	+ ·03	04·21	− ·01	04·20	+ ·03	04·23	+ ·01	04·24	+ ·02	04·26	+ ·02	04·28	+ ·03	04·31	+ ·13
07·56	+ ·03	07·59	+ ·04	07·63	− ·02	07·61	+ ·05	07·66	− ·04	07·62	·00	07·62	+ ·07	07·69	+ ·13
09·15	+ ·04	09·19	·00	09·19	+ ·07	09·26	+ ·04	09·30	+ ·05	09·35	− ·09	09·26	− ·07	09·19	+ ·20
36·35	+ ·02	36·37	+ ·02	36·39	·00	36·39	+ ·02	36·41	− ·01	36·40	+ ·05	36·45	+ ·03	36·48	+ ·13
43·80	+ ·02	43·82	·00	43·82	+ ·01	43·83	+ ·06	43·89	− ·02	43·87	+ ·01	43·88	+ ·05	43·93	+ ·13
Mean diffs.	+ ·03		+ ·01		+ ·01		+ ·03		·00		+ ·02		+ ·01		+ ·14

TABLE II.

Temperature-shift of Chlorine Lines, Ultra-violet Region.

Low Temperature.					High Temperature.						
Lunt.	Eder and Valenta.		Lunt.		Lunt.					Exner and Haschek.	
A 295(30) Cl in $GeCl_4$ Capillary. Short Exp. No Capacity.	(*a*) 10–20 mm.	(*b*) 30–40 mm.		A 291(20) A 292(20) Cl in $GeCl_4$ Capillary.	A 293(30) $SnCl_4$ Capillary.		A 291(30) A 292(30) Cl in $GeCl_4$ Capillary.			KCl Spark.	
		(1).	(2)-(1).	(2).	(3).	(3)-(2).	(4).	(4) (3).	(4)-(2).	(5).	(5) (2).
		3798·99	− ·03	98·96	99·12	+ ·16	99·31	+ ·19	+ ·35	99·6	+ ·64
		3805·38	− ·01	05·37	05·58	+ ·21	05·72	+ ·14	+ ·35	05·6	+ ·23
		09·70	− ·02	09·68	09·80	+ ·12	10·16	+ ·36	+ ·48	10·2	+ ·52
		20·40	+ ·03	20·43	20·60	+ ·17	20·83	+ ·23	+ ·40	20·7	+ ·27
		27·80	+ ·02	27·82	28·02	+ ·20	28·21	+ ·19	+ ·39	28·3	+ ·48
3833·50	3833·50	33·51	+ ·10	33·61	33·76	+ ·15	34·03	+ ·27	+ ·42	34·1	+ ·49
43·34	43·39	43·40	+ ·03	43·43	43·59	+ ·16	43·82	+ ·23	+ ·39	44·0	+ ·57
45·49	45·55	45·56	+ ·03	45·59	45·94	+ ·21	46·18	+ ·24	+ ·45	46·2	+ ·47
45·79 *	45·83	45·83	+ ·04	45·87							
45·92											
51·12 †	51·17	51·17	+ ·01	51·18	51·62	+ ·23	51·75	+ ·13	+ ·36	51·8	+ ·41
51·50	51·53	51·54	+ ·06	51·60							
		55·00	− ·07	54·93	55·13	+ ·20	55·38	+ ·25	+ ·45		
60·97 †	61·01	61·01	+ ·16	61·13	61·26	+ ·13	61·48	+ ·22	+ ·35	61·6	+ ·47
61·12											
		68·84	− ·04	68·80	68·94	+ ·14	69·16	+ ·22	+ ·36	69·2	+ ·40
		84·05	− ·04	84·01	84·24	+ ·23	84·55	+ ·31	+ ·54		
	14·06	3914·11	·00	14·11	14·29	+ ·18	14·57	+ ·28	+ ·46	14·40	+ ·29
	16·83	16·87	·00	16·87	17·05	+ ·18	17·28	+ ·23	+ ·41	17·2	+ ·33
	17·72	17·76	+ ·03	17·79	17·94	+ ·15	18·07	+ ·13	+ ·28		
	Mean diffs.		+ ·02			+ ·18		+ ·23	+ ·40		+ ·43

* Triple. † Double.

Description of Plate 4.

Strip I. Negative A 278 (20). Upper spectrum—Iron comparison, Exp. 8^s at beginning, and 8^s at end of chlorine exposure. Lower spectrum—Low temperature chlorine, residual chlorine in germanium tetrachloride tube, capillary of Geissler form, after deposition of major part of germanium, Ge 4227 still present, lines almost as sharp as the iron lines. Hg 4359 faint. 6-in. coil, 1 plate condenser, full self-induction, no gap. Exp. 8^m.

Strip II. Negative A 278 (50). Two consecutive exposures on same tube undisturbed. Upper—Residual chlorine as before. Exp. 6^m. Lower—6-in. coil, 2 plates, no self-induction, very small gap. Ge 4227 and many Cl lines show no shift, seven prominent Cl lines 4234 to 4277 show pronounced shifts. The pair 4259·62, 4261·31 widened owing to shift of red component only. Exp. 2^m 40^s.

Strip III. Negative A 278 (40). Upper—(Part of) iron comparison, Exp. 8^s beginning and end. Lower—Residual chlorine as in lower spectrum of strip II., same conditions except gap larger, temperature evidently higher, as certain lines are enhanced and the shifts are larger than in strip II. (see measures).

Strip IV. Negative A 284 (30). Upper—Iron comparison, two exposures of 8^s. Lower—Capillary of similar tube containing $SnCl_4$, pressure about 12 mm., direct discharge, 6-in. coil, no capacity, no self-induction, no gap. High temperature shown by diffuseness and shift of Cl lines, and by presence of the enhanced tin line 4330. Note faintness of Cl 4227 (almost coincident with Ge line) as compared with Eder and Valenta's intensities. Hg 4359 absent. Exp. 25^m.

Strip V. Negative A 259 (20).—Consecutive exposures, undisturbed end-on tube of germanium tetrachloride, pressure 10 mm. Upper—Exp. 5^m, 6-in. coil, direct discharge, no capacity, no self-induction, no gap. Mercury-break motor, second speed (higher) double number of contacts. Lower—tremendously brighter discharge, Exp. 8^s, 6-in. coil, 1 plate, no self-induction, small gap. Motor slowest speed, half number of contacts. Note the strongly shifted lines 4234–4277 and the absence of visible shift in many other lines. The high temperature is evidenced by the presence of the enhanced germanium line 4261. Hg 4359—strong in upper spectrum—has vanished, together with many chlorine lines.

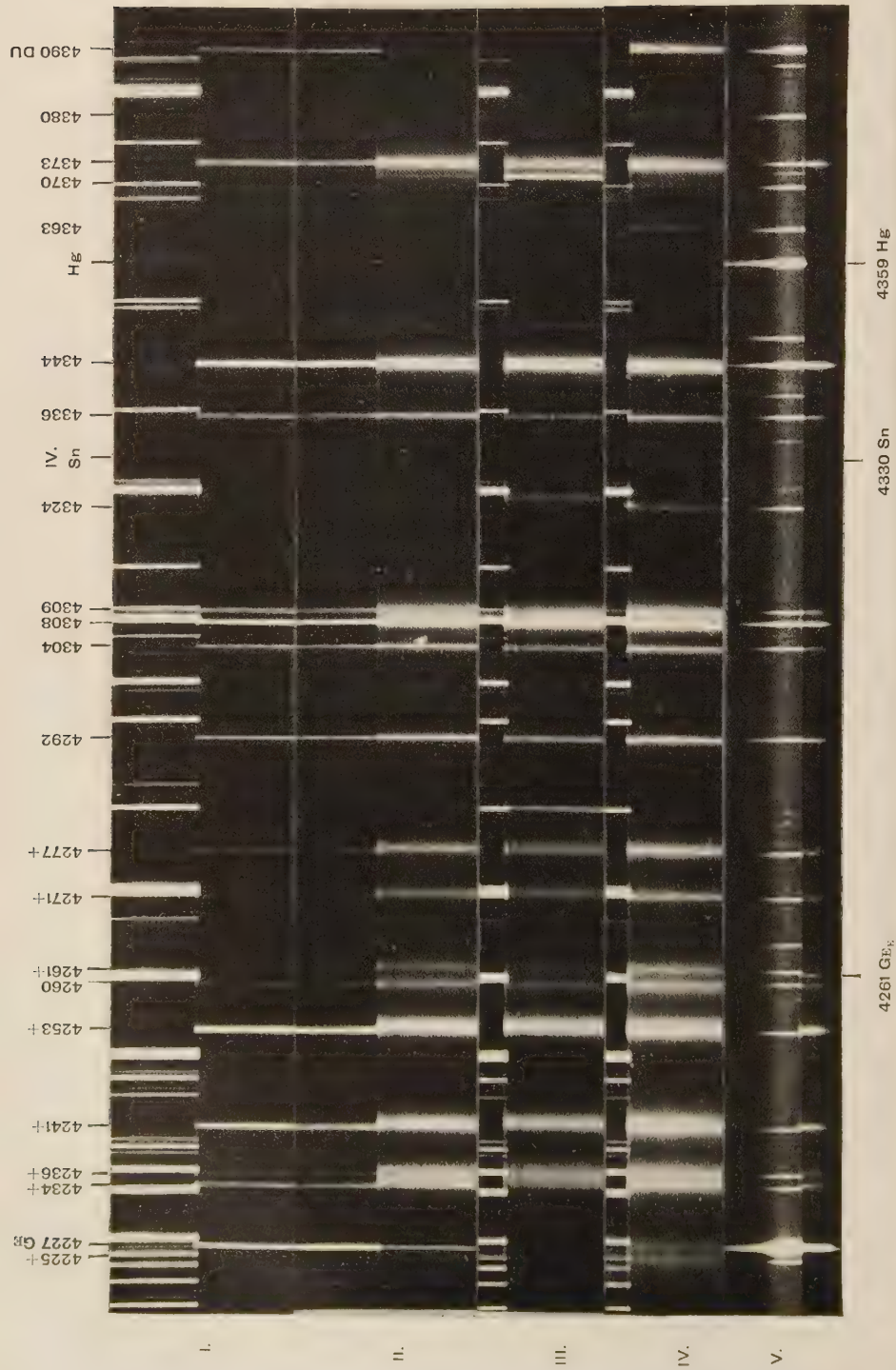
Description of Plate 5.

Strip I. Negative 291 (20). Upper—Iron comparison. Exp. 8^s, beginning and end. Lower—Residual Cl in $GeCl_4$ tube, capillary of Geissler form. Exp. 15^m, 6-in. coil, 1 plate, full self-induction, no gap, low temperature conditions. Motor slowest speed, half number of contacts.

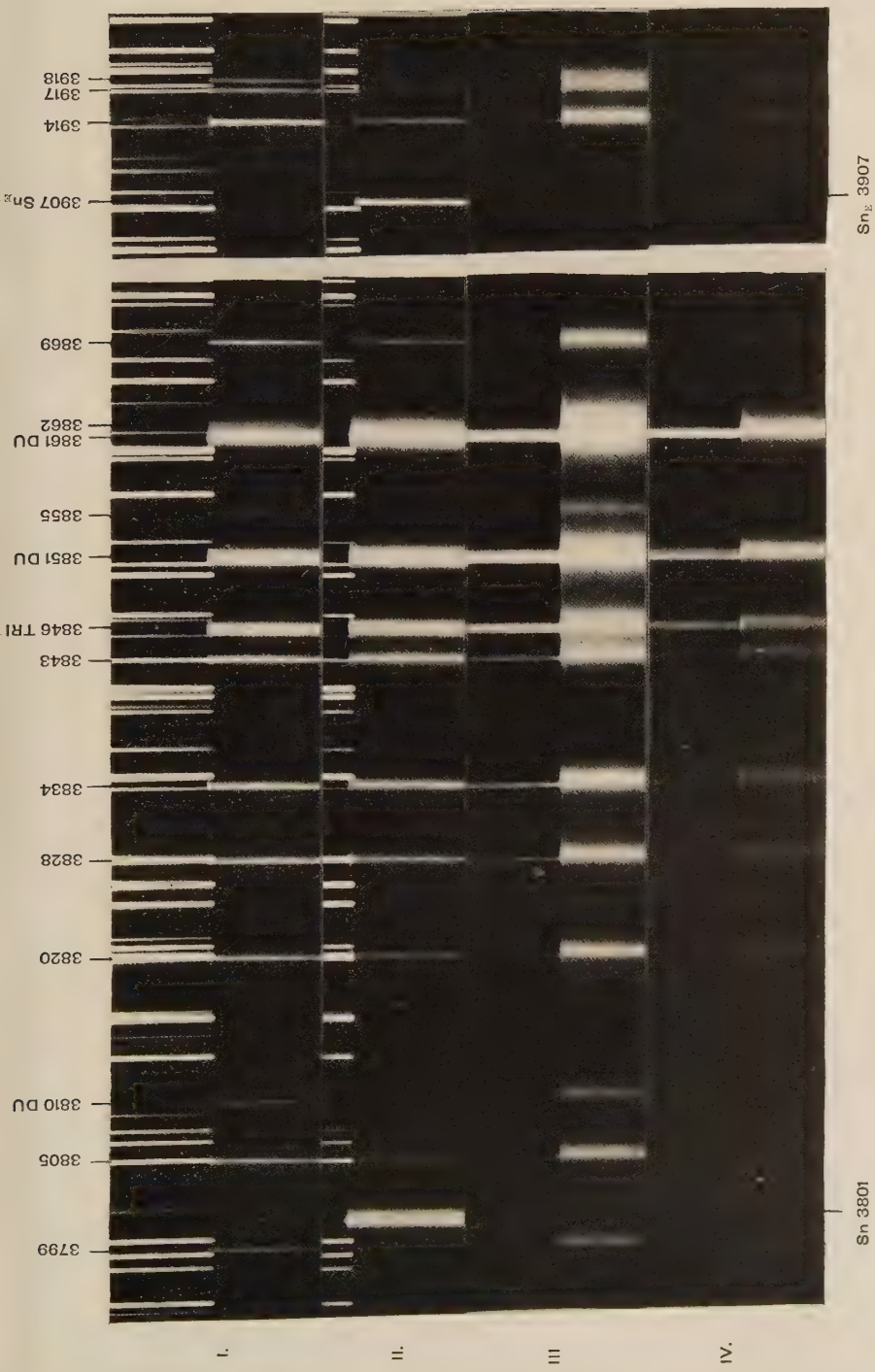
Strip II. Negative 293 (30). Upper—Iron spectrum as before. Lower—$SnCl_4$ tube, capillary. Exp. 10^m, 6-in. coil, no capacity, no self-induction, no gap. High temperature evidenced by presence of enhanced tin line 3907, Cl lines show an intermediate displacement (see measures). A tin spark, with self-induction, shows a much stronger 3801 line, but no 3907 enhanced line.

Strip III. Negative 291 (30). Upper—Residual chlorine in $GeCl_4$ tube. Exp. $7^m·5$ at beginning and end, same conditions as in strip I. (low temperature). Lower—same tube undisturbed. Exp. 10^m, 6-in. coil, 2 plates, no self-induction, small gap.

Strip IV. Negative 292 (30). Repeat of previous conditions with shorter exposures. Upper—4^m. Lower—2^m.



Temperature-shift of Chlorine Lines.


3799
3805
3810 DU
3820
3828
3834
3843
3846 TRI
3851 DU
3855
3861 DU
3862
3869
3907 Sn
3914
3917
3918
Sn 3801
Sn 3907
I.
II.
III.
IV.

On the Relation of Mean Parallax to Proper Motion and Apparent Magnitude. By Gorakh Prasad, M.Sc.

(*Communicated by the Astronomer Royal for Scotland.*)

The relation between mean parallax, proper motion, and apparent magnitude has been studied by several investigators. Kapteyn and van Rhijn * showed that

$$\log \bar{\pi} = -0{\cdot}690 - 0{\cdot}0713m + 0{\cdot}645 \log \mu \quad . \quad . \quad (1)$$

Luyten † has pointed out that the numerical constants entering in this formula are such that the latter may be replaced, practically without any error, by a simpler relation of the form

$$\bar{M} = a + b\mathrm{H} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a and b are constants and

$$\mathrm{H} = m + 5 + 5 \log \mu \quad . \quad . \quad . \quad . \quad (3)$$

The discussion by Luyten ‡ of such stars from the list of spectroscopic parallaxes for 1646 stars given by Adams, Joy, Strömberg and Burwell § as had also a known trigonometrical parallax shows that the relation (2) is not at variance with facts when a and b are made dependent on spectral class. Luyten does not consider values of H much greater than 12. Using 325 stars having $\mu \geqq 0''{\cdot}5$ and 181 stars having $0''{\cdot}3 \leqq \mu < 0''{\cdot}5$, and treating them together, irrespective of their spectral types, Seares ‖ comes to the conclusion that the relation of $\bar{M}$ to H is not linear, although a linear formula can be used on either side of the value $\mathrm{H} = 12{\cdot}5$.

Thus Luyten and Seares both agree that up to $\mathrm{H} = 12{\cdot}5$ the relation between H and $\bar{M}$ is linear. The present investigation was undertaken to obtain improved values of the constants in relation (2) by using the trigonometrical parallaxes of a much larger number of stars and treating them by an improved method. This has been done. The probable errors of the values of the constants as obtained in this paper are much smaller than the corresponding probable errors of the values found by Luyten. The investigation shows, moreover, that for the F and G stars the relation between H and $\bar{M}$ is not linear.

1. *Formula for calculating the Mean Absolute Magnitude.*

The method employed by Luyten consists in grouping the stars with respect to H. The absolute magnitude of every star is calculated from its observed parallax, and then the median values of the absolute magnitudes and H are derived for every group and used to solve the equa-

* *Mount Wilson Contr.*, No. 188; *Astroph. Journ.*, 52, 23, 1920.
† *Lick Obs. Bull.*, 11, 39 (No. 345), 1923.
‡ *Ibid.*, 10, 135 (No. 336), 1922.
§ *Mount Wilson Contr.*, No. 199; *Astroph. Journ.*, 53, 13, 1921.
‖ *Mount Wilson Contr.*, No. 273; *Astroph. Journ.*, 59, 310, 1924.

tions (2). Luyten points out that the use of the median value has the advantage of allowing stars with negative parallaxes to be used, for the absolute magnitude of any such star may be put down as $-\infty$. Seares simply takes the arithmetic means of H and the absolute magnitudes, no negative parallaxes being used. Obviously the plan of taking the mean of the calculated absolute magnitudes cannot be used with the material under discussion, for so many of the trigonometrical parallaxes come out as negative. In order to make use of these negative parallaxes more effectively, we shall take the arithmetic mean of the measured parallaxes π_0, the groups being formed with respect to H as usual, and calculate $\bar{M}$ by the following formula due to Strömberg*:

$$\bar{M} = 5 - 5 \log \overline{10^{-0 \cdot 2m}} + 5 \log \bar{\pi}_0 \quad . \quad . \quad . \quad (4)$$

The advantage of using this formula instead of the median is considered further on.

2. *Material Used.*

The present discussion is based on the trigonometrical parallaxes determined by photography at the Allegheny, McCormick, Yerkes, Greenwich, Sproul, Mount Wilson, and Dearborn Observatories. Out of the parallaxes published up to the present time, about 2200 in number, some relate to stars of which the spectral type is not known. Others could not be included in the present investigation, because they relate to stars whose total proper motion is not known. The available number of stars and parallax determinations in the case of each spectral type is shown in columns 2 and 3 of Table I. In cases where there are two or more determinations for the same star, the simple arithmetic mean of the various values has been taken as the parallax of the star. No regard has been paid to the probable errors of the parallaxes in forming the mean, because even the various determinations with fairly small probable errors are often very discordant. No parallaxes have been discarded on account of their large negative values or on account of their large probable errors. A few of the parallaxes have probable errors of 0″·020 or more. These might have been rejected, but in view of the fact that in general the true probable errors are greater than the published values and the latter are themselves affected with accidental errors, it has not been done. In the case of binaries the components have been treated as distinct stars. This has been done because in some of the binaries the companion belongs to a spectral type different from that of the primary; but only the common proper motion has been used in calculating H. The magnitudes used are all visual and on the Harvard scale. In the case of double and binary stars, if the separate magnitudes are not given in the Henry Draper Catalogue, these have been calculated from the combined magnitude as given in this catalogue and the difference of magnitude as given by Burnham. The spectral types used are those given in the Henry Draper Catalogue. In the case of double and binary stars, where the separate spectra were not given in H.D., the spectrum of the companion was assumed to be

* See *Astroph. Journ.*, 57, 296.

the same as that of the brighter star if the difference in magnitude was not more than 1, the combined magnitude was not fainter than 7·0, and the combined spectrum was not marked as peculiar. This has not been done very often; for example, in the case of the G stars this had to be done in seven cases only. For some stars the Mount Wilson estimated spectrum was used, the H.D. spectrum not being available. Such cases are few in number. Thus in the case of type G only three stars (companions to bright stars) were taken to be of type G on the authority of the Mount Wilson spectrum. In the case of the variable stars the mean magnitude has been used to calculate H and M, if the variation in apparent magnitude is less than one magnitude; otherwise the star has not been included in the discussion. The proper motions are generally from Boss or *Cincinnati Publication*, No. 18; but for the Greenwich stars, if the proper motion could not be derived from these sources, the total proper motion was calculated from the components (those derived from catalogue positions) published with the parallaxes. Only in the case of A stars eight of the proper motions had to be taken from various sources as given in Shorr's *Eigenbewegungs Lexikon*. Stars with $\mu < 0''\cdot005$ were excluded.

Van Maanen and Miss Wolfe * have studied the systematic differences in the parallaxes determined at different observatories. Some of the corrections deduced by them are pretty large. For the Greenwich parallaxes the correction amounts to $0''\cdot012$, the probable error of this value of the correction being only $0''\cdot001$. Strömberg † also finds for the observatories he considers similar corrections. Still, these corrections were not applied to the parallaxes used in this paper, because of the comparative scantiness of the material used in the investigation of Van Maanen and Miss Wolfe and the greater uncertainty of the corrections derived by Strömberg. Moreover, the only effect which these systematic errors can have is to cause a slightly greater dispersion in the values of M. To convert relative parallaxes into absolute ones, a constant reduction of $0''\cdot002$ has been applied to the parallaxes determined at the Mount Wilson Observatory, and $+0''\cdot005$ to the parallaxes determined at the other observatories.‡

3. *Relation between* H *and* M.

The stars of each spectral type were divided into a number of groups, each group except either the first (that with the smallest value of H) or the last (that with the largest value of H) containing the same number of stars. If the total number of stars exceeded an exact multiple of the number of groups by less than five, the extra number of stars was included in the first group; otherwise the last group contained a smaller number of stars than the other groups. The mean values of H and of M as calculated from formula (4) for these groups are shown in Tables II. to X., and plotted in figs. 1 to 6. In the case of the F and G stars for the range of values of H from 1 to 12

* *Mount Wilson Contr.*, No. 189.
† *Ibid.*, No. 220.
‡ *Ibid*, No. 199, p. 7.

and 1 to 14 respectively, it was assumed that the relation between H and $\bar{M}$ is

$$\bar{M} = a + bH + cH^2 \quad . \quad . \quad . \quad . \quad (5)$$

In all other cases it was assumed that this relation is

$$\bar{M} = a + bH \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The values of a, b, c were found by the Method of Least Squares, and they are shown, together with their probable errors, in Table I. The corresponding curves or straight lines are shown by continuous lines in the figures. The columns headed M_c or M'_c in Tables II. to X. give the value of the mean absolute magnitude as calculated from equation (5) or (6) with the appropriate values of a, b, c. The residuals are also shown in the tables.

TABLE I.

The Constants in the Relation between H *and* $\bar{M}$.

Spectral Class.	Number of Stars.	Number of Parallaxes.	Number of Groups.	Range of the Values of H.	a.	b.	c.
B8, B9, A0, A2, A3, A5	190	257	5	0–7	−0·43 ± ·19	·407 ± ·14	
F0, F2, F5, F8	313	453	10	1–12	−0·92 ± ·14	·762 ± ·044	−·0223 ± ·0031
F0, F2, F5, F8	248	361	8	6–12	+0·90 ± ·19	·357 ± ·022	
G0, G5	382	517	10	1–14	−2·11 ± ·18	·919 ± ·053	−·0223 ± ·0037
G0, G5	304	412	8	7–14	+0·11 ± ·38	·469 ± ·036	
K0, K2 Giants	155	202	5	0–7	−0·35 ± ·09	·307 ± ·022	
K0, K2, K5 Giants	185	237	5	0–7	−0·59 ± ·14	·364 ± ·032	
K0, K2 Dwarfs	153	219	5	8–14	−3·57 ± ·51	·866 ± ·045	
K0, K2 Dwarfs	184	258	6	6–14	−4·04 ± ·31	·905 ± ·091	
K5, Ma, Mb, Mc Giants	80	103	6	−1–7	−1·84 ± ·23	·647 ± ·056	
Ma, Mb, Mc Giants	50	68	4	0–7	−1·41 ± ·27	·435 ± ·062	
K5, Ma, Mb, Mc Dwarfs	56	94	4	11–16	+0·57 ± 1·71	·601 ± ·39	

TABLE II.

A Stars.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - \bar{M}$.
1	0·88	0·22	−0·07	−·29
2	2·94	0·43	0·77	+·34
3	3·97	1·21	1·19	−·02
4	4·94	1·52	1·58	+·06
5	6·77	2·54	2·33	−·21

TABLE III.

F Stars.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$ from Curve.	$\bar{M}'_c$ from Straight Line.	$\bar{M}_c - \bar{M}$.	$\bar{M}'_c - \bar{M}$.
1	1·59	0·22	0·23		+ ·01	
2	4·97	2·29	2·32		+ ·03	
3	6·06	2·89	2·89	3·06	·00	+ ·17
4	6·79	3·50	3·23	3·32	− ·27	− ·18
5	7·34	3·38	3·47	3·52	+ ·09	+ ·14
6	7·84	3·51	3·69	3·70	+ ·18	+ ·19
7	8·32	3·97	3·88	3·87	− ·09	− ·10
8	9·00	4·17	4·14	4·11	− ·03	− ·06
9	9·87	4·28	4·44	4·42	+ ·16	+ ·14
10	11·94	5·07	5·02	5·16	− ·05	+ ·09

TABLE IV.

G Stars.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$ from Curve.	$\bar{M}'_c$ from Straight Line.	$\bar{M}_c - \bar{M}$.	$\bar{M}'_c - \bar{M}$.
1	1·15	− 1·04	− 1·09		− ·05	
2	4·99	1·79	1·92		+ ·13	
3	7·65	3·66	3·62	3·70	− ·04	+ ·04
4	8·79	4·03	4·25	4·23	+ ·22	+ ·20
5	9·37	4·94	4·54	4·50	− ·40	− ·44
6	9·91	4·78	4·81	4·76	+ ·03	− ·02
7	10·41	5·15	5·04	4·99	− ·11	− ·16
8	10·95	5·29	5·28	5·24	− ·01	− ·05
9	11·58	5·27	5·54	5·54	+ ·27	+ ·27
10	13·14	6·17	6·12	6·27	− ·05	+ ·10

TABLE V.

K0, K2 Giants.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - M$.
1	0·56	− 0·10	− 0·18	− ·08
2	3·02	0·48	0·58	+ ·10
3	4·23	0·79	0·95	+ ·16
4	5·15	1·38	1·23	− ·15
5	6·43	1·64	1·62	− ·02

TABLE VI.

K0, K2, K5 Giants.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - \bar{M}$.
1	0·49	−0·32	−0·41	−·09
2	2·77	0·43	0·42	−·01
3	4·09	0·58	0·90	+·32
4	5·05	1·35	1·25	−·10
5	6·34	1·83	1·72	−·11

TABLE VII.

K0, K2 Dwarfs.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$ from First Straight Line.	$\bar{M}'_c$ from Second Straight Line.	$\bar{M}_c - \bar{M}$.	$\bar{M}'_c - \bar{M}$.
1	6·43	1·64		1·78		+·14
2	8·25	3·40	3·57	3·43	+·17	+·03
3	10·12	5·58	5·19	5·12	−·39	−·46
4	11·23	6·10	6·15	6·12	+·05	+·02
5	12·05	6·66	6·87	6·87	+·21	+·21
6	13·75	8·36	8·34	8·40	−·02	+·04

TABLE VIII.

K5, Ma, Mb, Mc Giants.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - \bar{M}$.
1	−0·20	−2·15	−1·97	+·18
2	2·02	+0·03	−0·53	−·56
3	3·04	−0·37	0·13	+·50
4	4·20	+0·84	0·88	+·04
5	5·13	+1·86	1·48	−·38
6	6·40	+2·08	2·30	+·22

TABLE IX.

Ma, Mb, Mc Giants.

Mean Values of H and M.

Number	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - \bar{M}$.
1	0·24	−1·32	−1·31	+·01
2	3·24	−0·25	0·00	+·25
3	4·65	1·09	0·61	−·48
4	6·35	1·11	1·35	+·24

TABLE X.

K5, Ma, Mb, Mc Dwarfs.

Mean Values of H and M.

Number.	$\bar{H}$.	$\bar{M}$.	$\bar{M}_c$.	$\bar{M}_c - \bar{M}$.
1	11·64	7·80	7·57	− ·23
2	13·32	8·33	8·58	+ ·25
3	14·54	8·97	9·31	+ ·34
4	15·96	10·45	10·16	− ·29

The various special points relating to the different spectral types will now be briefly taken up.

A Stars.—In this division all the A stars and also B8 and B9 stars have been included. Three stars with exceptionally large values of H,

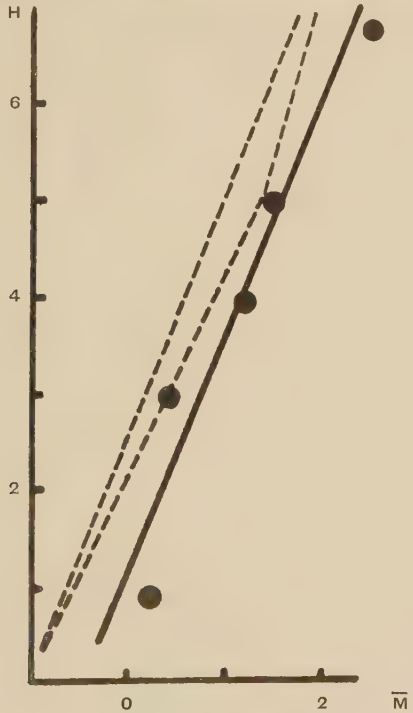


FIG. 1.—A stars.

viz. 9·3, 12·8, and 17·8, have been excluded. As the parallaxes and proper motions are relatively small for the A stars, the relation between H and $\bar{M}$ for them is more uncertain than the number of stars would lead us to expect. At first these stars were divided into ten groups, but the points thus obtained were more or less irregularly scattered. The points obtained by dividing the stars into five groups are more satisfactory, as can be seen from fig. 1. These stars were not investigated by Luyten.

F Stars.—This division includes all the F subdivisions, viz. F0, F2, F5, and F8. One star with an exceptionally large value of H, viz. H = 19·7, was excluded. The relation between H and $\bar{M}$ is seen from fig. 2 to be non-linear. This non-linearity is indicated by Luyten's work also, for the points for the F stars in fig. 1 of his paper * can be better represented by a curve than a straight line. The equation of the best straight line passing through the eight points with $\bar{H} > 6$ is also given in Table I. It is discussed below in connection with a similar straight line for the G stars.

G Stars.—This division includes both the subdivisions G0 and G5. Here also the relation between H and $\bar{M}$ is seen to be non-linear. The curve of the second degree shown in fig. 3 is seen to fit the points satisfactorily. A glance at fig. 2 of Luyten's paper will show that his points do not fit a straight line, or in fact any simple curve, so closely. This must be due to the smaller number (253) of the stars considered, to the inclusion of a number of the earlier and less reliable parallaxes, and to the use of the median instead of the mean value.

In the paper by Seares, no mean values of H less than + 5·9 have been used.† The relation between H and $\bar{M}$ for values of H from this up to 12·5 is found by him to be linear, as stated before. The first two groups of Table IV. were therefore omitted, values of H from 7·6 to 13·1 only being thus taken into consideration, and the equation of the best straight line through the remaining eight points was found. This is given in Table I. The regularity in the sign of the residuals (see Table IV.) shows that curvature is perceptible even in this short range. Moreover, the sum, 0·35, of the squares of the residuals is greater than the sum, 0·30, of the squares of the corresponding residuals for the curve. It appears, therefore, that the relation between H and $\bar{M}$ for the G stars should preferably not be taken as linear. A similar remark applies to the F stars. Here the residuals for the straight line passing through the eight points with the largest values of H do not show the same regularity in the signs of the residuals, but still the sum, 0·16, of the squares of these residuals is slightly greater than the corresponding sum, 0·15, for the curve.

K Stars.—This division includes only K0 and K2 stars. The points obtained are shown in fig. 4. It is evident that two straight lines will give a better representation than any curve of the second degree. It seems that the point H = 6·4, M = 1·6 lies on both the straight lines and could be included very well in the computations of both. Unlike the case of the M stars considered below, there is no sharp gap in the values of H to indicate a point of division. Eddington and Miss Douglas ‡ have taken M = 3·5 to be the dividing line between the K giants and dwarfs. From fig. 4 of this paper it is evident that the point H = 8·3, M = 3·4 should be included among the points for dwarfs. The point in question, with a value of M much less than 3·5, was therefore

* *Lick Obs. Bull.*, **10**, 137.

† It should be noted that the notation adopted in the present paper is that of Luyten's second paper (*Lick Obs. Bull.*, **11**, 39, No. 345). In the paper by Seares H stands for $m + \log_{10} \mu$.

‡ *Monthly Notices*, **83**, 115.

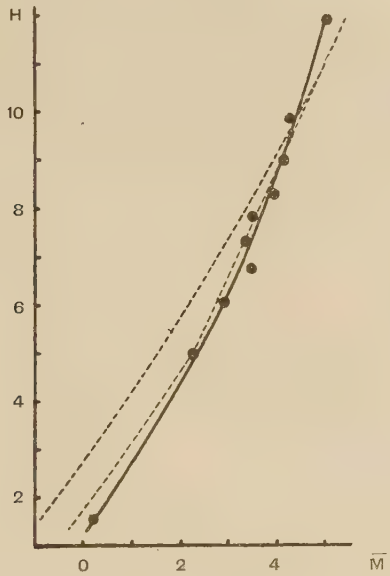


FIG. 2.—F stars.

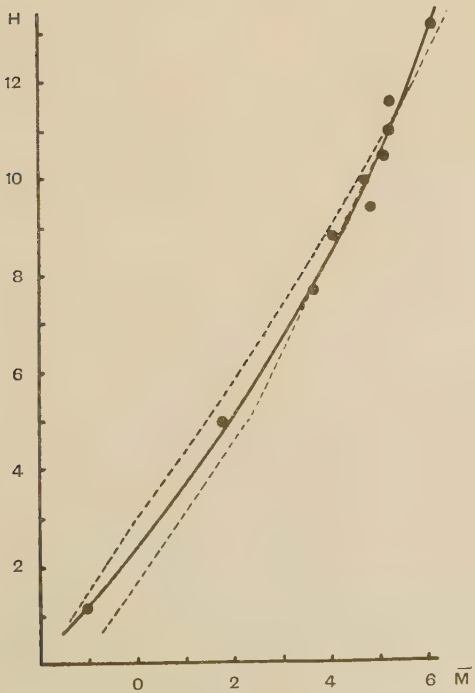


FIG. 3.—G stars.

included in the discussion of the giants only. But a second solution for the dwarfs has also been made with this point included. The equation of the straight line thus obtained, which is almost coincident with the former one, is given in Table I. and the residuals in Table VII. Luyten also has considered the giants and dwarfs separately, but his straight line for the dwarfs is very different from the line obtained here, as is shown in fig. 4.

M Stars.—This division includes K5 and M stars. Barnard's

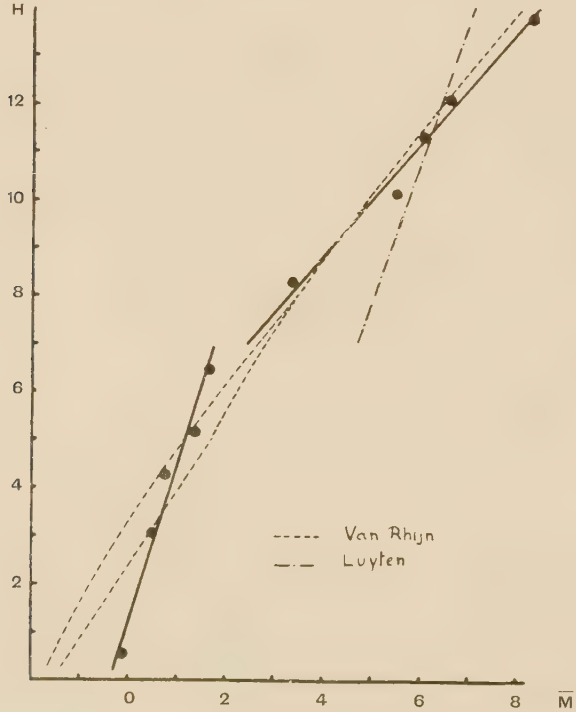


FIG. 4.—Ko and K2 stars.

Proper Motion star was excluded on account of the abnormally large value of its H. There is a sharp division between the giants and dwarfs marked by the absence of stars with a value of H between 8·1 and 10·2. The number of stars is not large enough to give a very satisfactory determination, and the probable errors of the constants are larger. In the case of the giants the straight line obtained for these stars differs considerably from the straight line obtained, as explained in Art. 8, from Van Rhijn's mean parallaxes for M stars. To see how far this difference was due to the inclusion of K5 stars, a second solution was made from which all the K5 stars were excluded. The straight line thus obtained is much nearer Van Rhijn's line, as is shown by fig. 5.

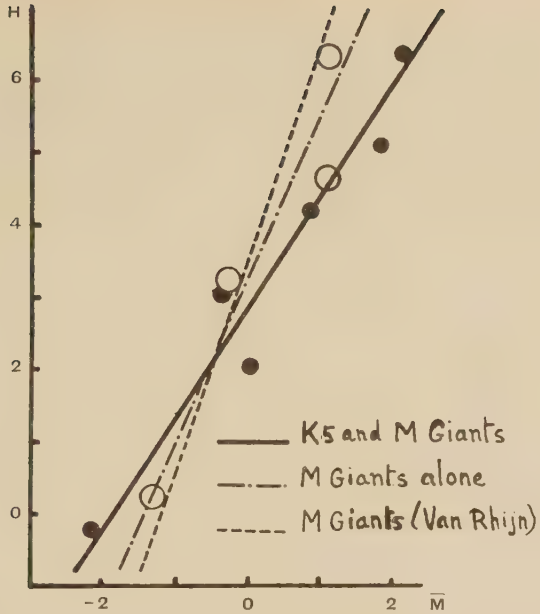


FIG. 5.—M Giants. The dots represent groups of K5 and M stars, the circles those of M stars alone.

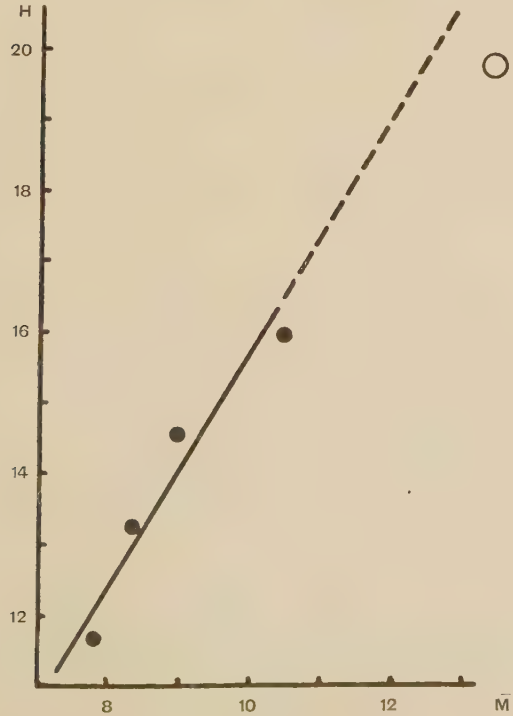


FIG. 6.—M Dwarfs. The circle represents Barnard's Proper Motion star.

In the case of the dwarfs the number of stars is smaller still, and consequently the relation between H and M is somewhat uncertain. It is interesting to note that Barnard's Proper Motion star lies fairly

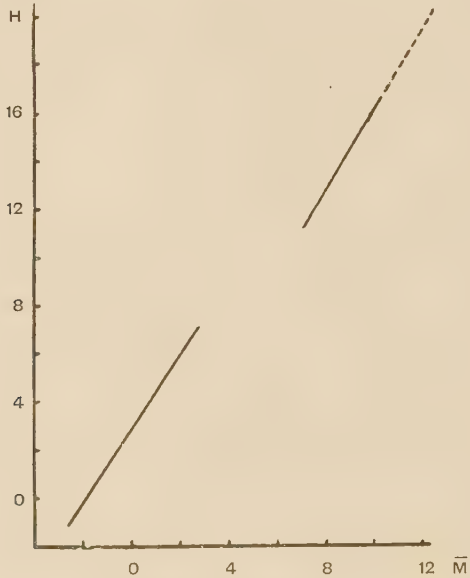


FIG. 7.—M stars.

near the straight line determined from the rest of the stars (fig. 6). The straight line for the giants is nearly parallel to that of the dwarfs, but the one is not a continuation of the other, as can be seen from fig. 7.

4. *Effect of Accidental Errors.*

It has been pointed out by Seares * that accidental errors in any quantity produce a systematic error in the mean of a number of values of this quantity if the frequency distribution of these values is not uniform. In view of this it seems necessary to examine the magnitude of the systematic errors in the mean values of H in the various groups. Fortunately, the accidental errors of the proper motions are much smaller than the accidental errors of the parallaxes, and therefore the errors in H are relatively unimportant. Since an error $\delta\mu$ in μ produces an error δH in H where approximately

$$\delta H = 2{\cdot}2\delta\mu/\mu,$$

the largest errors for each spectral type are likely to occur in group 1, which contains the smallest proper motions. There is another factor

* *Monthly Notices*, **84**, 15.

tending towards the same effect, namely, the great non-uniformity of the frequency distribution of the values of H within and in the neighbourhood of this group. Hence it will suffice to make sure that the systematic errors in such a group are negligible.

Since H is the same function of μ as M is of π, we find, almost exactly as in the case of absolute magnitudes discussed by me in a previous paper,* that the error in $\bar{\mathrm{H}}$ is

$$\frac{5\mathrm{N}h_1k^{-\frac{1}{2}}d\mathrm{H}\int_g^\infty\int_0^\infty u(\mu)e^{-h_1^2(\mu-\mu_1)^2}f(\mu_1,\mathrm{H})(\log\mu-\log\mu_1)d\mu d\mu_1}{\mathrm{N}h_1k^{-\frac{1}{2}}d\mathrm{H}\int_g^\infty\int_0^\infty u(\mu)e^{-h_1^2(\mu-\mu_1)^2}f(\mu_1,\mathrm{H})d\mu d\mu_1},$$

where N is the total number of stars considered,

h_1 is the modulus of precision of the proper motions,

k is the ratio of the circumference of a circle to its diameter,

dH is the difference between the largest and the smallest value of H in the group under consideration,

$u(\mu)$ represents the true frequency distribution of the proper motions, so that $\mathrm{N}u(\mu)d\mu$ is the number of stars having a true proper motion between μ and $\mu+d\mu$,

$f(\mu_1, \mathrm{H})$ is the number of stars which have their measured proper motion between μ_1 and $\mu_1+d\mu_1$ and their H (calculated from the measured proper motions) between H and H + dH divided by the total number of stars between the same limits of proper motion irrespective of their value of H,

and g is the smallest value of the measured proper motion considered.

We consider now an actual example, say group 1 of the G stars. Omitting the 5 stars with a measured proper motion of less than 0″·010 per year, assuming that no star with a true proper motion of less than 0″·010 per year occurs in this group (some such supposition is necessary, for if a single star with a true proper motion equal to zero were to be included in this group the error in $\bar{\mathrm{H}}$ would rise to infinity), and taking the probable error of the proper motions of this group to be 0″·006 per year, we find on computation that the error in $\bar{\mathrm{H}}$ is less than 0·01 in numerical value. We may conclude therefore that no appreciable systematic errors exist in the mean values of H.

In order to see if the very small proper motions had any harmful effect on the value of $\bar{\mathrm{H}}$ for this particular group, the values of $\bar{\mathrm{H}}$ and $\bar{\mathrm{M}}$ were computed after excluding (i) all the 5 stars with $\mu<0''\cdot010$, and (ii) all the 15 stars with $\mu<0''\cdot020$. The points corresponding to these groupings ($\bar{\mathrm{H}}=1\cdot37$, $\bar{\mathrm{M}}=-1\cdot13$) and ($\bar{\mathrm{H}}=2\cdot02$, $\bar{\mathrm{M}}=-0\cdot72$) lie fairly near the curve, the residuals being $+\cdot24$ and $+\cdot38$.

Though not connected with the present topic, it may be mentioned here that the last group was re-calculated by omitting the 6 stars with $\mu>2''\cdot00$, to find if the very large proper motions had any abnormal effect. The result is $\bar{\mathrm{H}}=12\cdot83$, $\bar{\mathrm{M}}=6\cdot06$. This gives a residual of $-\cdot04$ which is satisfactorily small.

Seares † states that "there is always danger of trouble in attempting

* *Monthly Notices*, **84**, 493. † *Ibid.*, **84**, 19.

to determine the relation of absolute magnitude to any other characteristic, x, even when x itself is free from error." While this is true when the groups are formed with respect to the absolute magnitudes, it is obvious that it is not true if the groups are formed with respect to x. Since the groups above have been formed according to H, the accidental errors of the parallaxes will produce no systematic errors in the mean absolute magnitudes. The apparent magnitudes, m, are used in the calculation of both H and M, and the accidental errors in m will produce systematic errors in both $\bar{\mathrm{H}}$ and $\bar{\mathrm{M}}$, but these must be very small.

5. *Systematic Errors.*

Since in formula (4) only $\bar{\pi}_0$ is used, the effect of the accidental errors in the parallaxes is much reduced. But the formula is not strictly true. If for each individual star the relation

$$\mathrm{M} = a + b\mathrm{H} + c\mathrm{H}^2 \quad . \quad . \quad . \quad . \quad (7)$$

were valid (and if the mean of the observed parallaxes π_0 were equal to the mean of the true parallaxes π, which may be assumed to be true for the purpose in hand), the value of $\bar{\mathrm{M}}$ given by (4) would be quite correct. This is easily seen by expressing π in terms of H and m from (7) and taking the logarithm of the mean. But equation (7) is not satisfied individually, and we are to calculate a, b, and c on the assumption that the relation $\bar{\mathrm{M}} = a + b\mathrm{H} + c\mathrm{H}^2$ holds for a group consisting of a large number of stars. We can estimate the error introduced by the use of (4) as follows:—

Consider any one group consisting of a large number, say n, of stars, and let $a + b\mathrm{H} + c\mathrm{H}^2$ have the value M for this group. Let the true values of the absolute magnitudes of the stars of the group be denoted by M′. Put

$$\mathrm{M} = \mathrm{M}' + x \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We suppose now that x has a normal frequency distribution, so that the number of stars for which x has a value between x and $x + dx$ is

$$nhk^{-\frac{1}{2}}e^{-h^2x^2}dx \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where k is the ratio of the circumference of a circle to its diameter as before. Then $\bar{\mathrm{M}}' = \mathrm{M}$, and the correction to be added to the mean absolute magnitude calculated from (4) is

$$\mathrm{M} - \{5 - 5 \log \overline{10^{-0\cdot2m}} + 5 \log \bar{\pi}\} \quad . \quad . \quad . \quad (10)$$

Substituting $10^{0\cdot2(\mathrm{M}-x-5-m)}$ for π, this reduces to

$$5 \log \overline{10^{-0\cdot2m}} - 5 \log \overline{10^{-0\cdot2x} \,.\, 10^{-0\cdot2m}}.$$

Now x and m can be supposed to be independent. This gives

$$\overline{10^{-0\cdot2x} \,.\, 10^{-0\cdot2m}} = \overline{10^{-0\cdot2x}} \,.\, \overline{10^{-0\cdot2m}},$$

and the correction (10) thus becomes

$$-5 \log \overline{10^{-0\cdot2x}}.$$

From (9) we have

$$\overline{10^{-0\cdot 2x}} = \frac{h}{k^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-h^2x^2}\, 10^{-0\cdot 2x}dx$$
$$= e^{0\cdot 01/h^2\mathrm{Mod}^2}.$$

Therefore the correction to the mean absolute magnitude calculated from (4) is, finally,

$$-\frac{0\cdot 05}{\mathrm{Mod}}\cdot\frac{1}{h^2} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This expression holds also when the parallaxes are considered to be affected by accidental errors and also if $c=0$ in (7).

Thus the error committed by the use of (4) is small if the dispersion is small, but a correction can be applied if necessary. The value of h is not easy to evaluate with certainty, and this is a disadvantage, but the use of the median of the individually calculated absolute magnitudes is also affected with a systematic error, as can be seen from what follows.

6. *Systematic Error of the Median Value.*

Consider the same group of stars as above, and let the absolute magnitudes calculated from the measured parallaxes π_0 be denoted by M_0. Put

$$M = M_0 + y \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The differences y between M and M_0 consist of two parts, one part, x, due to the dispersion of the true absolute magnitudes M' about M, and another part, say ξ, due to the errors in M_0 caused by the accidental errors in the parallaxes. Let

$$\pi = \pi_0 + z \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where the accidental errors z are supposed to be distributed normally, the modulus of precision of π_0 being denoted by h'. Since

$$M' = m + 5 + 5\log\pi \quad . \quad . \quad . \quad . \quad (14)$$

and

$$M_0 = m + 5 + 5\log\pi_0 \quad . \quad . \quad . \quad . \quad (15)$$

we have

$$\xi = M' - M_0 = 5\log(\pi/\pi_0) = -5\log(1 - z/\pi) \quad . \quad . \quad (16)$$

The frequency distribution of ξ in a sub-group consisting of all the stars having their true parallaxes between π and $\pi + d\pi$ is therefore represented by

$$0\cdot 2h'\pi k^{-\frac{1}{2}}(\mathrm{Mod})^{-1}e^{-h'^2\pi^2(1-10^{-0\cdot 2\xi})^2}\, 10^{-0\cdot 2\xi}d\xi \quad . \quad . \quad (17)$$

together with $n_1 p$ stars at $+\infty$, where

$$p = \frac{h'}{k^{\frac{1}{2}}}\int_{\pi}^{\infty} e^{-h'^2x^2}dx \quad . \quad . \quad . \quad . \quad (17a)$$

and n_1 is the number of stars in the sub-group. Thus

$$y = x + \xi \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where x is distributed according to (9) and ξ is distributed according to (17). The frequency distribution of y is therefore given by $\Phi(y)$, where *

$$\Phi(y) = \frac{1}{2k}\int_{-\infty}^{\infty} \Omega_1(\Theta)\Omega_2(\Theta)e^{i\Theta y}d\Theta \quad . \quad . \quad . \quad (19)$$

where

$$\Omega_1(\Theta) = \frac{h}{k^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-i\Theta x}e^{-h^2x^2}dx = e^{-\Theta^2/4h^2}$$

and

$$\Omega_2(\Theta) = \frac{c \cdot 2\pi h'}{k^{\frac{1}{2}}\,\mathrm{Mod}}\int_{-\infty}^{\infty} e^{-i\Theta\xi}e^{-h'^2\pi^2(1-10^{-0\cdot 2\xi})^2}10^{-0\cdot 2\xi}d\xi.$$

This is the distribution of ξ within the sub-group with a constant π. If the frequency distribution of π in the group is given by $f(\pi)$, where $f(\pi)$ can be calculated from the observed frequency distribution, $F(\pi_0)$, of the measured parallaxes π_0, by the formula †

$$f(\pi) = F(\pi) - \frac{1}{4h'^2}F''(\pi) + \frac{1}{32h'^4}F''''(\pi) + \ldots \quad . \quad (20)$$

the frequency distribution of y in the group is given by $\Psi(y)$, where

$$\Psi(y) = \int_0^{\infty} \Phi(y)f(\pi)d\pi \quad . \quad . \quad . \quad . \quad (21)$$

together with nq stars at $+\infty$, where

$$q = \int_0^{\infty} pf(\pi)d\pi \quad . \quad . \quad . \quad . \quad (21a)$$

The value of the median absolute magnitude found from the calculated absolute magnitudes M_0 is therefore $M - y_1$, where

$$\int_{-\infty}^{y_1} \Psi(y)dy = \int_{y_1}^{\infty} \Psi(y)dy + q \quad . \quad . \quad . \quad (22)$$

But the mean true absolute magnitude is M. Hence the error in the median is y_1, where y_1 is given by (22).

If the accidental errors in the parallaxes were absent, *i.e.* if all the ξ's were zero, the median would obviously coincide with the mean true absolute magnitude. Again, if the true absolute magnitudes, instead of being dispersed about the value M, were all equal to M, *i.e.* if all the x's were zero, then also the median would coincide with M. To see this we consider that the absolute magnitude is a monotone function of the parallax when m is kept fixed. This makes the median absolute magnitude correspond to the median parallax for a sub-group of stars with a constant m. Hence for every such sub-group, and therefore also for the whole group, the median absolute magnitude is not affected by any systematic error.

* Whittaker and Robinson, *The Calculus of Observations*, p. 170.
† *Ibid.*, p. 207.

It might appear at first sight that, since the median values of x and ξ are zero, the median value of y will also be zero, and hence that there will be no error in the median value of the absolute magnitudes M_0. But this in general is not so, for the frequency distribution of ξ is asymmetrical. In order to find what is the amount of systematic error in the median in practical cases, a typical example is worked out below.

7. *A Numerical Example.*

We shall employ for this example the material used by Luyten. Consider the group of 27 G stars which have their H between 2·0 and 3·9. We shall first find the numerical value of h. For this purpose we take the spectroscopic absolute magnitudes and reduce each of them to the value 3·0 of H. Thus if for any particular star the spectroscopic absolute magnitude is M_1 and the value of H is H_1, we take

$$M'_1 = M_1 - 0{\cdot}65(H_1 - 3{\cdot}0) \quad . \quad . \quad . \quad (23)$$

as the value of the spectroscopic absolute magnitude which this star would have had if its H had been 3·0. Here +0·65 is the value of b in the formula (2) as given in Luyten's paper. The arithmetic mean of the absolute magnitudes thus obtained is −0·2, and the standard deviation of these absolute magnitudes from the mean is 0·81. The accidental errors of the spectroscopic absolute magnitudes may be supposed to be normally distributed with a probable error of 0·4,* *i.e.* with a standard deviation of 0·59. Hence the standard deviation of x in equation (18) must be $\sqrt{\{(0{\cdot}81)^2 - (0{\cdot}59)^2\}} = 0{\cdot}56$, which gives $h = 1{\cdot}26$. The probable error of the trigonometrical parallaxes will be assumed to be 0″·010, giving $h' = 47{\cdot}7$.

Out of the 27 parallaxes 5 are negative, 7 are in the neighbourhood of 0″·005, 12 are in the neighbourhood of 0″·025, and 3 are greater than 0″·050. With the above value of the probable error 12 stars at $\pi = 0''{\cdot}005$ and 15 stars at $\pi = 0''{\cdot}025$ will give an observed distribution of parallaxes somewhat similar to the above, the number of negative parallaxes being precisely 5. For the sake of convenience, therefore, we assume that the true distribution of parallaxes is 12 values of $\pi = 0''{\cdot}005$ and 15 values of $\pi = 0''{\cdot}025$.

With the value of h derived above the frequency-distribution (9) gives ·340 n, ·245 n, ·075 n, and ·010 n stars between $M \pm 0{\cdot}25$ at $M = 0{\cdot}0$, $\pm 0{\cdot}5$, $\pm 1{\cdot}0$, $\pm 1{\cdot}5$, and $\pm 2{\cdot}0$ respectively. In view of the difficulty of evaluating the integrals occurring in (19), the simple assumption will here be made that there are ·340 n, ·245 n, ·075 n, and ·010 n stars for which x has the values 0·0, $\pm 0{\cdot}5$, $\pm 1{\cdot}0$, and $\pm 1{\cdot}5$ respectively. The distribution (17) for $\pi = 0''{\cdot}025$ requires 4·9, 11·6, 16·9, 16·9, 13·5, 9·7, 6·7, and 19·8 per cent. of the ξ's to lie between $-\infty$ and $-1{\cdot}5$, $-1{\cdot}25 \pm {\cdot}25$, $-0{\cdot}75 \pm {\cdot}25$, $-0{\cdot}25 \pm {\cdot}25$, $+0{\cdot}25 \pm {\cdot}25$, $+0{\cdot}75 \pm {\cdot}25$, $1{\cdot}25 \pm {\cdot}25$, and between $+1{\cdot}5$ and $+\infty$ respectively, the last group including the stars to which the absolute magnitude $-\infty$ is assigned on account of a negative parallax. To find

* *Astroph. Journ.*, 53, 27.

$\Phi(y)$ at, say, $y = 0{\cdot}25$, we notice that the stars for which y has a value lying between $0{\cdot}25 \pm {\cdot}25$ consist of all the stars for which $x = -1{\cdot}5$ and ξ lies between $1{\cdot}75 \pm {\cdot}25$, or $x = -1{\cdot}0$ and ξ lies between $1{\cdot}25 \pm {\cdot}25$, . . . or $x = 1{\cdot}5$ and ξ lies between $-1{\cdot}25 \pm {\cdot}25$. Hence the number of such stars must be $n({\cdot}010 \times {\cdot}045 + {\cdot}075 \times {\cdot}067 + {\cdot}245 \times {\cdot}097 + \ldots + {\cdot}010 \times {\cdot}116)$, where $4{\cdot}5$ is the percentage of stars for which ξ lies between $1{\cdot}75 \pm {\cdot}25$. We can thus find $\phi(y)$, but since we want merely to know for how many stars y is positive, the numerical work is considerably shortened. We find thus that $0{\cdot}520\ n$ stars have a positive y and the rest have a negative y. If the median value of y were zero, the number of stars having a positive y would be not $0{\cdot}520\ n$, but $0{\cdot}500\ n$. The difference corresponds to an error in the median of $0^{M}{\cdot}053$.

Similarly the error in the absolute magnitude of a group of stars, every one of which has a parallax of $0''{\cdot}005$, is found to be $0^{M}{\cdot}071$.

Finally, the error in the median of the group of stars, 15 of which have a true parallax of $0''{\cdot}025$ and 12 a true parallax of $0''{\cdot}005$, is found to be $0^{M}{\cdot}055$.

Formula (11) shows that the error in the absolute magnitude calculated from the mean parallax is $0^{M}{\cdot}073$. Thus there is not much to choose between the two different methods on the score of the smallness of the systematic errors caused by the accidental errors in the parallaxes. But, as is well known, the probable error of the median is 25 per cent. greater than that of the arithmetic mean, and for this reason the use of formula (4) is preferable.

No correction was applied to the mean absolute magnitudes shown in Tables II. to X.

8. *Comparison with the Results of Van Rhijn.*

It is interesting to compare the relation between H and $\bar{M}$ arrived at here with that deducible from the mean parallaxes found by Van Rhijn* from an exhaustive study of measured parallaxes, proper motions, and radial velocities. Stars are selected for parallax measurement on account of their brightness or large proper motion, and the influence of this selection on the relation between H and $\bar{M}$ is also clearly brought out by such a comparison. The last sections of tables 34 to 38 of Van Rhijn give the mean parallaxes of stars of various spectral types of determined proper motion and apparent magnitude. From each of these we can construct another table giving the mean absolute magnitude when H is known. The Tables XI. to XV. have thus been obtained, the entries being mean absolute magnitudes and the arguments H and m. They show that the mean absolute magnitude is not a one-valued function of H. To know the mean absolute magnitude definitely we must know also the apparent magnitude (or the proper motion).

* "On the Mean Parallaxes of Stars of determined Proper Motion, Apparent Magnitude, and Galactic Latitude for each Spectral Class," *Groningen Publications*, No. 34, 1923.

TABLE XI.

Mean Absolute Magnitude of A Stars of Determined H *and* m, *calculated from Table* 34 *of Van Rhijn.*

H \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
− 3	−2·4								
− 2	−2·1	−1·9							
− 1	−1·6	−1·6	−1·4						
0	−1·0	−1·1	−1·1	−0·9					
1	−0·4	−0·6	−0·7	−0·5	−0·3				
2	0·2	−0·1	−0·2	−0·2	−0·1	0·2			
3	0·9	0·5	0·3	0·2	0·3	0·4	0·7		
4	1·6	1·2	0·8	0·7	0·6	0·7	0·9	1·2	
5	2·3	1·8	1·5	1·2	1·0	1·0	1·2	1·4	1·7
6	3·0	2·5	2·0	1·7	1·5	1·4	1·5	1·7	1·9
7	3·7	3·1	2·7	2·2	1·9	1·8	1·8	1·9	2·1
8		3·8	3·3	2·8	2·5	2·2	2·2	2·2	2·3
9			3·9	3·4	3·0	2·7	2·5	2·5	2·6
10				4·0	3·5	3·2	3·0	2·9	2·8
11					4·1	3·7	3·4	3·3	3·1
12						4·2	3·9	3·6	3·5
13							4·3	4·0	3·9
14								4·4	4·2
15									4·6

TABLE XII.

Mean Absolute Magnitude of F Stars of Determined H *and* m, *calculated from Table* 35 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
− 3	−3·1								
− 2	−2·8	−2·4							
− 1	−2·4	−2·1	−1·7						
0	−2·0	−1·7	−1·4	−0·9					
1	−1·4	−1·2	−1·0	−0·6	−0·2				
2	−0·7	−0·6	−0·4	−0·2	0·2	0·5			
3	0·1	0·1	0·2	0·3	0·5	0·9	1·2		
4	1·0	0·8	0·8	0·9	1·0	1·3	1·6	1·8	
5	1·9	1·6	1·5	1·5	1·6	1·8	2·0	2·3	2·4
6	2·7	2·4	2·2	2·2	2·2	2·3	2·5	2·7	2·9
7	3·7	3·2	3·0	2·8	2·7	2·8	2·9	3·2	3·4
8	4·6	4·1	3·7	3·5	3·4	3·4	3·5	3·6	3·8
9	5·5	4·9	4·5	4·2	4·0	3·9	3·9	4·1	4·3
10	6·3	5·7	5·2	4·9	4·7	4·5	4·5	4·6	4·7
11		6·5	5·9	5·6	5·3	5·1	5·1	5·0	5·2
12			6·7	6·2	5·9	5·7	5·6	5·5	5·6
13				6·9	6·5	6·2	6·1	6·0	6·0
14					7·1	6·8	6·6	6·5	6·5
15						7·3	7·1	7·0	6·9
16							7·6	7·4	7·3
17								7·8	7·7
18									8·1

TABLE XIII.

Mean Absolute Magnitude of G Stars of Determined H *and* m, *calculated from Table* 36 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
− 3	−3·1								
− 2	−2·9	−2·4							
− 1	−2·6	−2·8	−1·7						
0	−2·1	−1·8	−1·4	−0·9					
1	−1·5	−1·3	−1·0	−0·6	−0·2				
2	−0·8	−0·7	−0·4	−0·2	0·2	0·5			
3	0·0	0·0	0·1	0·3	0·6	0·9	1·2		
4	0·8	0·7	0·7	0·9	1·1	1·3	1·7	1·8	
5	1·7	1·5	1·4	1·5	1·6	1·8	2·1	2·3	2·5
6	2·6	2·3	2·1	2·2	2·2	2·4	2·5	2·8	3·0
7	3·5	3·1	2·9	2·8	2·8	2·9	3·1	3·3	3·5
8	4·4	3·9	3·6	3·5	3·4	3·5	3·6	3·8	4·0
9	5·3	4·8	4·4	4·2	4·1	4·1	4·1	4·3	4·5
10	6·2	5·5	5·2	4·9	4·7	4·7	4·6	4·7	4·9
11		6·4	5·9	5·6	5·3	5·2	5·2	5·3	5·4
12			6·6	6·2	5·9	5·8	5·7	5·8	5·9
13				6·9	6·5	6·3	6·3	6·2	6·3
14					7·2	6·9	6·8	6·7	6·7
15						7·4	7·2	7·2	7·2
16							7·7	7·6	7·5
17								8·0	8·0
18									8·4

TABLE XIV.

Mean Absolute Magnitude of K Stars of Determined H *and* m, *calculated from Table* 37 *of Van Rhijn.*

H. \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
− 3	−3·0								
2	−2·8	−2·4							
− 1	−2·4	−2·2	−1·8						
0	−2·0	−1·8	−1·5	−1·1					
1	−1·5	−1·3	−1·2	−0·9	−0·5				
2	−0·9	−0·8	−0·7	−0·5	−0·2	0·1			
3	−0·2	−0·2	−0·1	0·0	0·2	0·4	0·7		
4	0·5	0·5	0·5	0·6	0·7	0·8	1·1	1·4	
5	1·3	1·2	1·2	1·3	1·3	1·4	1·5	1·7	2·0
6	2·0	2·0	2·0	2·0	1·9	2·0	2·1	2·2	2·4
7	2·8	2·8	2·8	2·7	2·7	2·7	2·7	2·8	2·8
8	3·6	3·6	3·5	3·5	3·5	3·4	3·4	3·4	3·5
9	4·4	4·4	4·4	4·3	4·3	4·2	4·2	4·1	4·1
10	5·1	5·1	5·1	5·1	5·0	5·0	4·9	4·9	4·8
11		5·9	5·9	5·9	5·8	5·8	5·8	5·7	5·6
12			6·7	6·6	6·6	6·6	6·5	6·5	6·4
13				7·4	7·4	7·4	7·3	7·3	7·2
14					8·2	8·2	8·1	8·1	8·1
15						8·9	8·9	8·9	8·9
16							9·7	9·7	9·6
17								10·4	10·4
18									11·2

TABLE XV.

Mean Absolute Magnitude of M Stars of Determined H *and* m, *calculated from Table* 38 *of Van Rhijn.*

H \ m	2.	3.	4.	5.	6.	7.	8.	9.	10.
− 3	−2·1								
− 2	−1·8	−1·8							
− 1	−1·6	−1·5	−1·5						
0	−1·2	−1·2	−1·2	−1·1					
1	−0·9	−0·9	−0·9	−0·9	−0·8				
2	−0·5	−0·6	−0·6	−0·6	−0·5	−0·5			
3	−0·2	−0·2	−0·2	−0·2	−0·2	−0·2	−0·2		
4	0·2	0·2	0·1	0·1	0·1	0·2	0·2	0·2	
5	0·5	0·5	0·5	0·5	0·5	0·5	0·5	0·4	0·5
6		0·9	0·9	0·9	0·9	0·8	0·8	0·8	0·7
7			1·2	1·2	1·2	1·2	1·2	1·2	1·2
8				1·6	1·6	1·6	1·5	1·5	1·5
9					2·0	1·9	1·9	1·9	1·9
10						2·4	2·3	2·2	2·2
11							2·7	2·7	2·6
12								3·0	3·0
13									3·5

These tables show that for the smaller values of H the range in $\bar{M}$ for a given value of H is in most cases considerable. But several of the entries in these tables have no meaning in practice. Consider, for example, Table XIII. for the G stars. There are in the sky only two G stars brighter than apparent magnitude 2·4. Hence entries under $m = 2$ can be neglected. Again, the blank space in the lower left-hand corner represents the omission of stars with $\mu > 4''\cdot00$ per year. The lowest entries correspond to $\mu = 4''\cdot00$ and the entries immediately above them to $\mu = 2''\cdot50$. On account of the scarcity of stars with such large motions we may neglect these entries also. A few more of the mean absolute magnitudes for the bright stars with very large proper motions may similarly be neglected. Only two G stars with $m > 9\cdot4$ are included in the present discussion, and therefore we may neglect the entries under $m = 10$. The blank space in the upper right-hand corner represents the omission of stars with $\mu < 0''\cdot010$. The top entries correspond to stars with $\mu = 0''\cdot010$. These also can be neglected. The extreme curves showing the relation between H and $\bar{M}$ as given by the remaining entries in the table are shown in fig. 3 by the dotted lines. We see that our curve lies between the two extreme curves obtained from Van Rhijn's results. The curve on the right corresponds to the fainter stars ($m = 5$ to 9) of small proper motion ($\mu = 0''\cdot020$) up to the point $H = 5$, $M = 2\cdot3$, and to stars of apparent magnitude 9 only beyond this point. We see that as more and more parallaxes of the fainter stars with small proper motions are measured, the lower part of the curve giving the relation between H and $\bar{M}$ derived from all the measured parallaxes will shift more and more to the right.

The agreement in the case of the G stars between the line obtained

in this paper and the line obtained from Van Rhijn's results is exceptional. In the case of the A, F, and K stars the lines of Van Rhijn are more to the left for those portions of the figure which correspond to the smaller values of M and therefore to the smaller values of π. This can be seen from figs. 1, 2, and 4, the dotted lines denoting the lines of Van Rhijn. This indicates that the smaller trigonometrical parallaxes are on the average relatively larger than the mean parallax derived from parallactic motions and similar considerations, a point which has been noted by Van Rhijn * also. The disagreement between the line obtained in this paper for the K5 and M giants and Van Rhijn's line for the M giants has been mentioned before. The line for the M stars alone agrees fairly well with Van Rhijn's line, but, unlike the case of the A, F, and K stars, the smaller trigonometrical parallaxes in this case are slightly smaller than the mean parallaxes of Van Rhijn. The disagreement in the lines for the K stars is very marked. In order to see if the inclusion of the K5 stars among the K stars will produce better agreement, a solution was made for all the K0, K2, and K5 stars together. The line thus obtained practically coincides with the line for the K0 and K2 stars. The disagreement obviously is not due to this cause. The absolute magnitudes of the M stars with a value of H greater than 10, as obtained from Van Rhijn's parallaxes, is quite different from the absolute magnitudes obtained here. Evidently his parallaxes do not apply to the dwarfs.

In conclusion, I wish to express my gratitude to Professor Sampson for his kind help and encouragement.

Summary.

1. The relation between $H \equiv 5 + m + 5 \log \mu$ and mean absolute magnitude has been found for each spectral type separately. The result is shown in Table I.

2. The discussion is based upon the trigonometrical parallaxes. The negative parallaxes have not been rejected.

3. The mean absolute magnitude has been calculated from the mean parallax. The advantage of this procedure, the systematic error to which it gives rise, and the effect of the accidental errors of the parallaxes and proper motions are considered.

4. The relation between H and mean absolute magnitude is found to be linear except for the F and G stars. In the case of the K and M stars giants and dwarfs had to be considered separately.

5. Comparison with the mean parallaxes published by Van Rhijn shows agreement in the case of G stars. In the case of A, F, and K stars the smaller trigonometrical parallaxes are in the mean greater than the mean parallaxes of Van Rhijn. In the case of M stars there is fair agreement, the mean of the smaller trigonometrical parallaxes being slightly less than the corresponding mean parallaxes of Van Rhijn.

Royal Observatory, Edinburgh:
1924 *December* 5.

* *Loc. cit.*, p. 61.

Fig. 1. June 25 λ = 220°

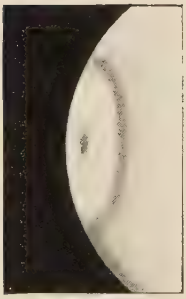


Fig. 2. June 30 λ = 169°

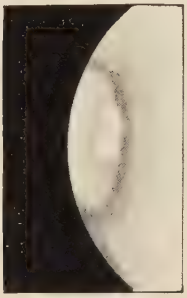


Fig. 3. July 15. λ = 11°

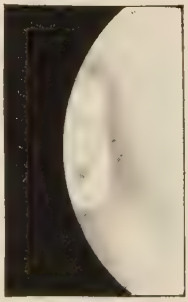


Fig. 4. July 19. λ = 334°

Fig. 5. Aug 22. λ = 17°

Fig. 6. Sept 22. λ = 85°

Observations of Mars in 1924. By Rev. T. E. R. Phillips, M.A. (Plates 6, 7, 8.)

The unusually close approach of Mars during the past year has caused special attention to be given to the study of its surface features by planetary observers. The following notes summarise briefly the observations of the writer at Headley with the Coleman 8-inch equatorial refractor by Cooke lent to him by the Society and a 12¼-inch Calver equatorial reflector. The powers employed on both telescopes ranged from about 230 to 450 diameters. A few notes were also made at the 18-inch reflector belonging to the British Astronomical Association.

The planet was in opposition on 1924 August 23, but was actually at its minimum distance from the Earth on the previous day, when its diameter attained a value of 25″·1. Unfortunately, however, it was very poorly placed in the sky for English observers, its declination at opposition being rather over 17°·5 south; and, in addition to this grave disadvantage, the meteorological and seeing conditions proved exceedingly unfavourable throughout nearly the whole period of the observations, especially after the end of July. Certain parts of the disc were never seen well at all, *e.g.* the *Syrtis Major* was not once observed near the central meridian of the planet under really satisfactory conditions. Between June 23 and November 30 twenty-five whole-disc drawings were made (six of which are reproduced on the accompanying plate) together with some sketches of particular features. A few transit observations of particular points on the surface for longitude and some micrometer measures of the diameter of the south polar cap were also secured. In this paper the usual nomenclature has been adopted, and the terms "sea," "canal," "lake," "snow," etc., which are in almost universal use, have also been employed for descriptive purposes.

1. *The South Polar Regions.*—Throughout the period covered by the observations the latitude of the centre of the disc varied between −16° and −24°, so that the South Pole was continuously presented to view. The vernal equinox of the southern hemisphere occurred on May 12 and the summer solstice on October 6; so that the circumstances would have been specially favourable—had the weather been good—for a study of the melting of the south snow-cap and closely related phenomena.

On several occasions after the latter part of June dark rifts were observed in the cap and one very dark spot, while the white area showed some striking irregularities in brightness. Six sketches showing typical appearances of the cap are reproduced (Plate 6), from which it will be seen that some specially bright portions were observed which presented the aspect of projecting lobes. One of these, near longitude 300°, was probably *Novissima Thule* of Schiaparelli, and two others in longitude 30° ± and 100° ± respectively were very conspicuous. The centre of the cap is clearly not coincident with the pole of rotation, as is indicated by its appearance in figs. 4 and 5 of Plate 7. The direction of the displacement seems to be along the meridian of 40° ± and the amount to be of the order of 10° or rather less.

The melting of the snow was not very rapid during the earlier portion of the period covered by the observations, but it became considerably accelerated in August and September. By November 10 (Plate 7, fig. 6) the cap was found to be small, but subsequent observations indicate that the rate of melting became checked. It will be recalled that at the 1894 apparition (near the time of sun-spot maximum) the cap entirely disappeared, and on the whole its behaviour this year (near the time of spot minimum) seems consistent with the relationship to which Mons. E. M. Antoniadi drew attention in *M.N.*, **76**, 643, between the melting of the Martian polar caps and the cycle of solar activity.*

It has been shown by Professor W. H. Pickering that measures made by means of the filar micrometer of objects like a Martian snow-cap when it is small, and when it is not feasible to set the webs actually on it, are liable to considerable error, and need to be increased by approximately one-third of their value.† It is probable, however, that the errors of measurement are small when the cap is large, and the following direct measures were made between the end of July and the early part of September:—

TABLE I.

Date.	Measured Diameter of Cap.	Areocentric Arc.
	″	°
July 29	8·12	42
Aug. 10	7·94	38
Aug. 11	7·90	36
Sept. 2	3·47	16

These measures, though few, are sufficient to indicate the acceleration in the rate of melting which set in in August.

2. *The Maria.*—On the whole these seemed to the writer to be less dark than usual. It appeared at times as if they were partially veiled by haze in the Martian atmosphere. They became darker as the season advanced, but were noted as faint again between longitudes 40° and 90° on September 22 and 24.

Omitting certain lakes and small patches the markings showing the greatest intensity were apparently *Sinus Sabæus*, *Mare Tyrrhenum*, *Hellespontus*, *Auroræ Sinus*, and *Mare Sirenum*. The *Syrtis Major* was perhaps less dark than usual. A feature which appeared particularly enfeebled was *Pandoræ Fretum*. There was a dark spot near its *p* end, but throughout the greater part of its length it was narrow and faint, and quite different in aspect from its appearance in 1909, when it was broad and dark. It also seemed to the writer that the coast-line of *Hellespontus* had retreated towards the east, and fig. 3 in Plate 7 shows

* In *M.N.*, **82**, 244, a note by the Secretaries draws attention to a similar conclusion arrived at by M. Shajn of the Pulkovo Observatory. His paper had been published in Russian and was not generally known. The two investigations were entirely independent.

† *H.A.*, **32**, 117. See also *B.A.A.J.*, **29**, 220.

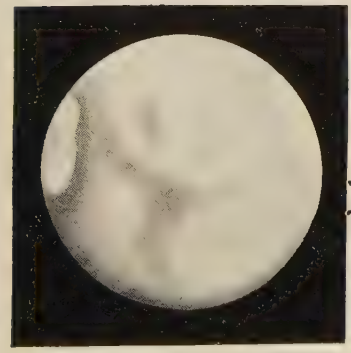


Fig. 1. July $13^{d}\ 14^{h}\ 30^{m}$ $\lambda=61^{\circ}$ $\phi=-18^{\circ}$

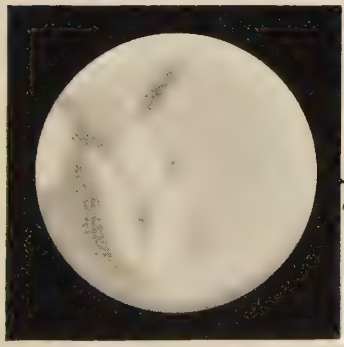


Fig. 2. Aug. $12^{d}\ 12^{h}\ 0^{m}$ $\lambda=110^{\circ}$ $\phi=-17^{\circ}$

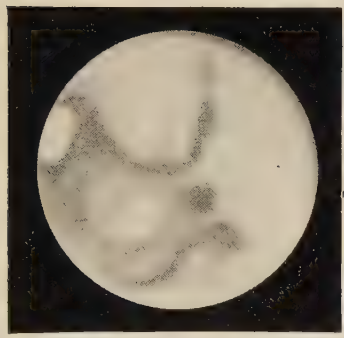


Fig. 3. Sept. $2^{d}\ 13^{h}\ 40^{m}$ $\lambda=309^{\circ}$ $\phi=-16^{\circ}$

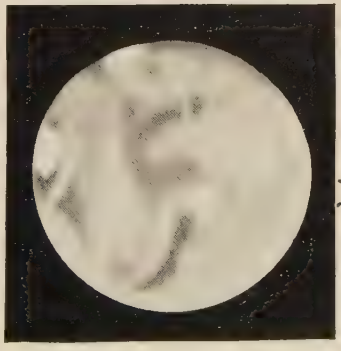


Fig. 4. Sept. $27^{d}\ 9^{h}\ 30^{m}$ $\lambda=23^{\circ}$ $\phi=-18^{\circ}$

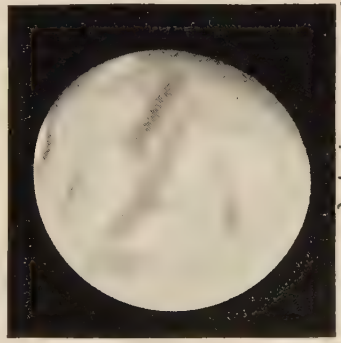


Fig. 5. Oct. $13^{d}\ 8^{h}\ 30^{m}$ $\lambda=220^{\circ}$ $\phi=-17^{\circ}$

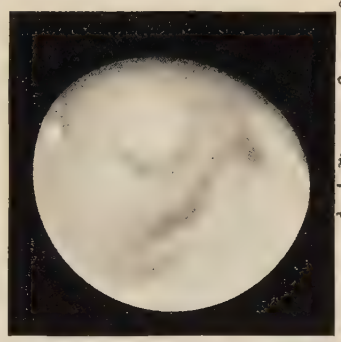


Fig. 6. Nov. $10^{d}\ 6^{h}\ 0^{m}$ $\lambda=277^{\circ}$ $\phi=-23^{\circ}$

S. Sabæus and *Hellespontus* as a strong, continuous, curved streak with the included region almost a blank.

The following colour notes were made:—

July 13. *Auroræ S.* near central meridian. Light regions salmon-red. Dark areas brownish with tinge of green. 18-in. and 12¼-in. reflectors.

August 3. *M. Sirenum* near central meridian. A beautiful blue-green. 12¼-in. reflector.

August 22. *Margaritifer Sinus* near central meridian. A greenish tinge in the Maria. 8-in. refractor.

3. *Lighter Regions in the Maria and White Areas.*—These in general presented a normal aspect. *Hellas* on September 2 was seen to be much brighter in its northern portion, and to be divided across its centre by a broadish band (*Peneus*) (see Plate 7, fig. 3). *Solis Pons* and *Œnotria* were clearly visible in the Syrtis on the same night. *Libya* was usually shaded, but on September 2 it appeared rather bright when near the central meridian. It regained its dusky appearance, however, on approaching the limb a few hours later (Plate 7, fig. 3). Both *Eridania* and *Elysium* seemed rather less bright than usual. *Hesperia* and *Atlantis* were both shaded, and the latter was usually not very clearly defined. *Electris* was seen on October 13 to brighten considerably as it approached the limb, and *Ausonia* (*S.*) was comparable with the brightest part of *Hellas*. *Argyre* appeared as a bright (though not white) extension of *Noachis*, and the latter was observed to be less bright than *Deucalionis Regio*. *Thaumasia* was distinctly shaded, as also was *Pyrrhæ Regio*. A small light area was seen on *M. Sirenum* north of, but close to, *Gorgonum S.* on September 14 and October 13, and a bright narrow band crossed *Auroræ S.* from *Eos* to *Aurea Cherso* on September 27.

Regions which were observed to be particularly bright were the district north of *Atlantis*, *Æolis*, *Edom Promontorium*, the southern part of *Thymiamata*, *Aromatum Promontorium*, and a white area (perhaps not permanent) *f Solis Lacus*. On October 13 the bright region on *Æolis* seemed so extended as to blot out the *Cyclops*, *Cerberus II.*, and *Eunostos* canals, and to be bounded on the west by a rather broad streak which was continuous with the *Hyblæus* canal (Plate 7, fig. 5). When observed during the previous presentation on September 2 the *Cyclops* and *Eunostos* had been visible, though it was noted that the region further west was dusky and confused, and that the details near the tip of *M. Cimmerium* were not easy to define. A whiteness was observed at various times about the planet's north limb, but usually without any definite boundary.

4. *The Canals.*—Objects very diverse in character are included under this title. As, however, the term is almost universally applied to them it is here retained.

The canals observed at Headley in 1924 were in nearly all cases of a soft and diffuse nature. The strongest and darkest of those seen were *Cerberus*, *Ganges*, *Nectar*, *Amenthes*, and *Tartarus*. *Nepenthes-Thoth* was still fairly conspicuous, but appeared to fade during the latter

part of the period of observation. *Læstrygon*, *Antæus*, *Cyclops*, *Eunostos*, *Hyblæus*, *Chaos*, *Styx*, *Peneus*, *Euphrates*, *Gehon*, *Indus*, *Jamuna*, *Nilokeras*, *Agathodæmon*, *Orœ*, *Araxes*, *Sirenius*, *Gorgon*, and *Eumenides-Orcus* were all seen at some time or other together with *Xanthus* and *Scamander*. *Eumenides-Orcus* was merely a broad, diffuse, and very irregular shading, containing condensations which it was difficult to locate accurately. There was a somewhat conspicuous canal bounding the bright extension of *Æolis* on the west which connected *Hyblæus* with a point on the *M. Cimmerium* (Plate 7, fig. 5) and a canal not previously observed, which may possibly have been only a contrast effect at the boundary of the bright area on *Thymiamata*, appeared on September 27 to connect *Oxia Palus* with the *f* prong of *Furca*.

5. *The Lakes.*—The following were observed:—*Trivium Charontis*, *Pambotis Lacus*, *Oxia Palus*, *Niliacus Lacus*, *Tithonius Lacus*, *Solis Lacus*, *Phœnicis Lacus*, *Lacus Lunæ*, a lake (unidentified but fairly dark) on the canal *Tartarus*, some dark condensations along the border of *Thaumasia*, dark areas in the south portion of *Hellespontus* (*Depressiones Hellespontiæ*), and some vague spots in the diffuse shading of *Eumenides-Orcus* which have not been satisfactorily identified. *Lacus Mœris* was not distinguished from a swelling of the *Nepenthes*, but this region was not observed under really satisfactory conditions. *Juventæ Fons* was also missed. Definition was sufficiently sharp to show this minute object on one night, but the planet was getting very low and the surface details were dimmed by haze. *Coloe Palus*, *Ismenius Lacus*, and other northern lakes were not well placed on the disc for observation in 1924. *Niliacus Lacus* was much foreshortened but easily visible on September 27. On this same night *Oxia Palus* came out quite distinctly once or twice. *Lacus Tithonius* always appeared rather large and diffuse and was widely double. *Solis Lacus* with *Nectar* broad and strong presented a fig-shaped aspect. It seemed to the writer to be generally paler than usual, but was recorded as relatively dark on July 13. *Lacus Phœnicis* was seen only on August 12, when in the best moments it showed a small but quite dark (almost black) nucleus (Plate 7, fig. 2).

6. *Deformations of the Terminator.*—These were observed on two nights only, viz. September 27 and October 13, the two best nights of the apparition.

On September 27 the appearance was that of two depressions separated by a rather bright elevation or projection. The south depression, where the terminator crossed a dark region, was the more pronounced of the two, and the elevation may well have been due to a bright area *f*, the *Solis Lacus* seen on other occasions when well on the disc. It is probable, therefore, that the irregularities as observed were largely a contrast effect, and hence subjective; but on the other hand, the bright area may have been a cloud formation and the observed projection partly real. The appearance was first noted about $9^h\ 15^m$ and persisted till $9^h\ 45^m \pm$.

On October 13 the deformation was much more strongly marked and persisted for a much longer time. It was first noted by Mr. F. J. Hargreaves (who was observing at Headley that night) and by the writer independently at about $9^h\ 15^m$, and did not finally disappear till

$11^h\ 30^m \pm$, at which time the planet was very low in the sky. In this case there was again a bright elevation with depressions or indentations north and south of it, the latter one being especially strongly marked. The general appearance was as shown in the accompanying diagram (Plate 8, fig. 2).

It was at once concluded that the bright projection was due to the north portion of *Hellas* which had been observed as brighter than the rest of the island on September 2 (Plate 7, fig. 3). But by 11^h the terminator crossed the north part of *Hellespontus*, so the cloud formation, or whatever was responsible for the observed irregularities, must have extended westwards from *Hellas* right across *Yaonis Regio*. The phenomenon was a particularly interesting one, and it seems hardly possible to doubt that in this case at least the observed projection was real. The development of cloud in this region was in all probability the cause of the appearance.

7. *Transit Observations.*—The observations in Table II. were obtained for determining the longitudes of special points on the planet's surface. It had been intended to secure several transits of these and other features, but the exceedingly unfavourable conditions of weather and seeing made this impossible.

TABLE II.

Object.	Time of Passage across Central Meridian of Disc.				ω.
		d	h	m	°
Fastigium Aryn	Oct.	2	10	58	358·1
,, *f* prong of *Furca*	Aug.	25	12	15	359·0
Margaritifer Sinus	Aug.	22	11	45	18·2
Auroræ Sinus	Sept.	22	8	30	53·8
Solis Lacus (centre)	Sept.	22	10	50	87·9
p tip of *M. Sirenum*	Sept.	17	11	5	136·9
Titanum Sinus	Sept.	14	11	30	170·1
Hammonis Cornu	Aug.	31	12	40	312·1 } 310·3
,, ,,	Sept.	2	13	38	308·5 }

8. *Occultation of Mars by the Moon.*—Mars was occulted by the Moon on 1924 November 5, the angles of disappearance (dark limb) and reappearance (bright limb) from the north point, calculated for Greenwich, being 352° and 320° respectively. The following observations were obtained at Headley.

The air was thick and murky and thin clouds were passing. The Moon and Mars were both dim and very red.

Disappearance.—Clouds passing.

G.M.T.	
h m s	
8 1 24·2	Deformation of planet's limb first noticed.
8 3 21·4	Mars gone.

The planet seemed to have disappeared about a second before the time given, but a speck of light flashed out again. Since it was so nearly a grazing occultation a small portion of the planet's disc was probably glimpsed momentarily through an indentation in the Moon's limb.

Reappearance.—Moon and planet hidden by cloud. At $8^h\ 24^m\ 10^s$ G.M.T. a fleeting view was obtained when the planet appeared as a projection from the Moon's limb. It was estimated as about one-fifth of its diameter out. Nothing more was seen till rather over two minutes late when a gap of some 2″ or 3″ appeared to separate the limbs.

The 8-inch refractor was used and power 230.

The error of the Clock was determined from the Greenwich, Paris, and Nauen Wireless Time Signals.

Position of Observatory:

Longitude	. .	0° 16′ 2·2″	west.
Latitude	. .	51 16 32·3	north.

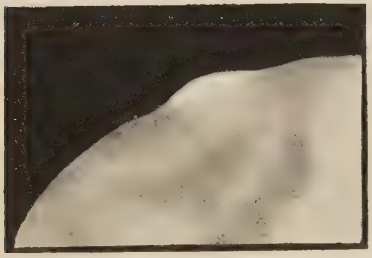
Headley Rectory, Epsom:
1924 *December* 9.

Addendum.—Since the above paper was written a few further observations have been made. The seeing was exceptionally good on December 10, and with high powers a considerable amount of detail could be made out. Plate 8, fig. 3, shows a bright "bridge" cutting across *Hellespontus* into *Noachis* from *Yaonis Regio*, which may well indicate a further development of the cloud-like area responsible for the irregularity of the terminator observed on October 13. The whitish district of *Hellas* also appeared to extend much further south than usual. Other features of interest were the recovery of *Pandoræ Fretum* and the increased duskiness of *Noachis*. A large white area was also visible about the north limb. A drawing of September 17 is also added to Plate 8 showing the white spot *f* the *Solis L.*, which is believed to have caused the apparent projection of September 27.

The seeing was again exceedingly fine on the evening of December 20, the planet's limb being perfectly sharp with a power of 670 on the 8-inch refractor. The disc, however, was practically *blank*, only the faintest ghosts of markings being visible notwithstanding that the *Mare Cimmerium* was crossing the central meridian! It seems as if the Martian features must have been concealed by haze or some other atmospheric obstruction. The south polar cap, too, was quite invisible, and it has not again been seen.

1924 Sept. 17d 11h 30m λ=143° φ = -17°

Fig. 1.



1924 Oct. 13d 9h 30m ...

Fig. 2.

1924 Dec. 10d 6h 30m λ - 353° φ 25°

Fig. 3.

Proper Motions in Latitude of Sunspots and Faculæ.

(*Communicated by the Astronomer Royal.*)

Sunspots.—An analysis of the apparent proper motions in latitude of sunspots forms, as is well known, the basis of a determination of the position of the Sun's axis, such as Carrington's and that published by the Royal Observatory, Greenwich, in 1913. In both publications, the possibility was necessarily considered of any real systematic drift in latitude being attributable to the spots themselves. Carrington remarked * on small apparent motions as follows: for latitudes 20°–40° N. and S. a poleward daily motion not exceeding 2′ (*i.e.* 0°·85 in a sidereal rotation of the Sun); 10°–20° N. and S., a feeble tendency equatorwards of about 1′ per diem, or 0°·4 per rotation; 10° N. and S. to the equator, no apparent systematic motion discernible from the available material. In a later table † he deduced "an average tendency towards the N. Pole of 90″ in the time during which the Sun rotates through 30 degrees," *i.e.* 0°·3 per rotation, but he believed this to be due to the omission of some small correction and not to actual proper motions. The magnitudes of the proper motions found by Carrington were not confirmed by the more extensive material dealt with in the Greenwich publication, but it was found that "a slight general tendency is suggested for high-latitude spots in either hemisphere to move away from the equator, but for those in lower latitudes to move towards it." ‡

Since a knowledge of any possible proper motions of sunspots is of importance to the theory of their structure, it seemed worth while to make independently a fresh analysis of the proper motions of long-lived spots, with the addition of the data from the solar cycle just completed. The period thus covered by the Greenwich measures is fifty years. If the latitude of a spot is compared in successive rotations when the longitude from the node is the same, any error due to a wrongly assumed position of the Sun's axis is thereby eliminated, and any difference of latitude may therefore be attributed to real proper motion. The "Recurrent" groups therefore afford good material for this purpose. Such groups are tabulated in the Greenwich *Catalogue of Recurrent Groups of Sunspots*, 1874–1906, and in subsequent yearly volumes of the *Greenwich Photoheliographic Results.* As supplementary data, the "Revival" groups have also been included in this discussion. These "Revivals" are groups of spots seen in consecutive rotations of the Sun, but with definite breaks in their history between each rotation. This latter feature excludes them from being classified as "Recurrent." Since the year 1917 they have been tabulated in the *Photoheliographic Results*, and for the purposes of this paper a list, based on the examination of the *Ledgers*, has been prepared from 1889 to 1916.

In dealing with the centres of Recurrent groups, it appeared possible that an apparent systematic proper motion might result from some characteristic in the development of the spot groups. About

* *Observations of Spots on the Sun*, by R. C. Carrington, F.R.S., p. 222.
† *Ibid.*, p. 242.
‡ *M.N.R.A.S.*, **73**, 687.

60 per cent. of all groups are of the type designated in the Greenwich publications as "stream," or "stream of normal type," and in the Mount Wilson publications as "bipolar" groups. A large number of these are seen in the first rotation as a fully developed stream and in the second as a single regular spot, which is, in most cases, the leader which has survived. If streams were usually parallel to the solar equator, the latitude would thereby not be systematically affected, but the axes of most streams of this type are inclined to the solar equator, the leader having the smaller latitude.* The differences in latitude between the leader and the *centre* of the stream were tabulated for about 170 typical groups. The average difference is 0°·3—a quantity which is large enough to introduce a small apparent systematic proper motion equatorwards, if a large number of groups of this type are included.

In consequence, it was thought advisable to compare in successive rotations the latitudes of the components rather than the centres of such groups, wherever the components were sufficiently long-lived and suitable for measurement.

The material has therefore been divided up as follows:—

(1) Recurrent single spots and components of groups seen in two rotations. [Years 1874–1923.]
(2) Recurrent groups, excluding those whose components have been dealt with in (1). [Years 1874–1923.]
(3) Revival groups, *i.e.* groups which are seen in successive rotations, but where there is a clear break in the continuity of spot activity. [Years 1889–1923.]

Very frequently the development of the Revival group is apparently independent of the earlier activity in the region from which it arises; at other times, however, the renewed activity is localised near the position of the preceding, and sometimes of the following, components of the earlier group of spots. But, on the whole, the objection against using indiscriminately the centres of the Recurrent groups does not apply to the Revival groups; moreover, there is no other way in which the measures can be utilised, owing to the brief but complete discontinuity of the spot activity. Although for some reasons the Revival groups should be considered of less weight than the Recurrent groups, they give very consistent results, besides providing independent evidence of any appreciable systematic drifts in latitude.

1. *Recurrent Single Spots.*—Reference may be made to *M.N.R.A.S.*, **72**, 565, for the details of the comparison of the daily measures of latitude of spot groups, whereby any existing proper motions may, in a simple manner, be freed from any error due to a wrongly assumed position of the Sun's axis, which is known, however, to be very small. It may be mentioned that the unit of reduction for heliographic latitude from the micrometer measures is one-tenth of a degree of solar latitude. The majority of spots in this class are of the "regular" type, presenting little difficulty of accurate measurement. The results derived from the single spots are tabulated below: a *plus* sign indicates a poleward drift; a *minus* sign an equatorward drift.

* *Contributions from Mount Wilson Observatory*, **165**, 15.

TABLE I.

Proper Motions in Latitude of Single Spots per Sidereal Rotation.

Hemisphere.	Latitudes 0°–10°. No. of Spots.	Random P.M.	Systematic P.M.	Latitudes 10°–20°. No. of Spots.	Random P.M.	Systematic P.M.	Latitudes 20°–30°. No. of Spots.	Random P.M.	Systematic P.M.
		°	°		°	°		°	°
North	74	1·00	+ ·08 ± ·10	135	1·02	− ·04 ± ·09	38	0·99	+ ·24 ± ·13
South	91	0·86	− ·15 ± ·08	131	0·90	+ ·15 ± ·07	30	0·91	+ ·03 ± ·14
N. and S.	165	0·92	− ·06 ± ·06	266	0·96	+ ·05 ± ·05	68	0·95	+ ·14 ± ·10

In the above classification no systematic drifts are evident. The three zones have therefore been combined with results as follows:—

Mean systematic proper motion . . .	+ 0°·03 ± ·04
Mean random ,, ,, . . .	± 0·94
Number of observations	499

A further classification was then made according to the phase of the solar cycle—increasing to maximum; decreasing to minimum. There was no indication that the proper motions were related to the phase, as the following figures show—increase, + 0°·01; decrease, + 0°·04.

2. *Recurrent Groups.*—As explained in the introductory note, these measures relate to the estimated centres of gravity of groups as a whole. Where the components were sufficiently distinctive and long-lived to be treated individually, they have been classified in Table I., and the measures of the group-centre have been excluded from this section. Table II. is therefore derived from entirely independent data.

TABLE II.

Proper Motions in Latitude of Groups of Spots per Sidereal Rotation.

Hemisphere.	Latitudes 0°–10°. No. of Groups.	Random P.M.	Systematic P.M.	Latitudes 10°–20°. No. of Groups.	Random P.M.	Systematic P.M.	Latitudes 20°–30°. No. of Groups.	Random P.M.	Systematic P.M.
		°	°		°	°		°	°
North	81	1·33	− ·44 ± ·12	139	1·17	− ·06 ± ·08	28	1·49	+ ·21 ± ·23
South	68	1·16	− ·10 ± ·12	140	1·28	− ·30 ± ·09	54	1·20	+ ·08 ± ·14
N. and S.	149	1·25	− ·29 ± ·08	279	1·23	− ·18 ± ·06	82	1·30	+ ·13 ± ·12

3. *Revival Groups.*—Although groups of this class are numerous, if comparisons in latitude are to be made strictly at intervals of a sidereal rotation, the number becomes limited on account of the gaps in their history which is characteristic of their type. The error of the position of the Sun's axis is, however, a small quantity, which it is sufficient to eliminate approximately in dealing with this somewhat rougher material. The difference in mean latitude of the Revival group in the first and second rotation, together with the corresponding interval in days, has accordingly been used to derive the proportionate proper motion in twenty-five days, provided the observed interval differed only a few days from the sidereal rotation period.

TABLE III.

Proper Motions in Latitude of Revival Groups per Sidereal Rotation.

Hemisphere.	Latitudes 0°–10°. No. of Groups.	Random P.M.	Systematic P.M.	Latitudes 10°–20°. No. of Groups.	Random P.M.	Systematic P.M.	Latitudes 20°–30°. No. of Groups.	Random P.M.	Systematic P.M.
North	122	1°·23	+·02°±·09	178	1°·20	+·10°±·08	26	1°·13	+·06°±·20
South	78	1·16	+·05±·11	193	1·33	+·26±·08	50	1·35	−·20±·16
N. and S.	200	1·20	+·03±·07	371	1·27	+·18±·06	76	1·27	−·11±·12

Referring to the results in Tables I.–III., if they are considered as supplementary, which it certainly seems legitimate to do, then the evidence is against any systematic drift in latitude, as much as one-tenth of a degree per sidereal rotation. The mean random proper motions of single recurrent spots is 0°·96, and for both the Recurrent and Revival groups 1°·25 per rotation. Considering the three tables separately, Table I., representing the most accurate measures it has been possible to collect, does not establish or even suggest a systematic drift; Table II. shows in zones 0°–10°, 10°–20° consistent *minus* signs; whilst Table III. shows rather smaller numbers but with *positive* signs for the same zones. The figures have been examined in detail for each zone according to the cycle. They are not smooth enough to be entirely conclusive, and from other considerations which follow it is not clear that at present they can be regarded as other than accidental.

A tendency for the direction of a drift of a spot to persist for several weeks has sometimes been noted, but this does not appear to be general. From a tabulation of the proper motions of single spots or groups of spots observed in more than two consecutive rotations, it is seen that although a constant drift is sometimes maintained, just as often the direction is changed after the second rotation. As opposed to this, however, it is found that a certain rotation or year of a cycle will yield a "run" of signs* from widely different parts of the Sun's photosphere, but individually they do not persist. The Recurrent groups in Cycle IV., tabulated in Table IV., show, for instance, a marked run of *minus* signs, but this does not extend to the Revivals. It seems, therefore, justifiable to treat the "runs" as accidental, but a more detailed examination will be desirable when observations are accumulated from additional cycles. In this connection a comparison of proper motions of single spots and groups with their corresponding magnetic polarities will be a useful piece of work when sufficient details are available from Mount Wilson.

TABLE IV.

Cycle.	Years.	Single Spots. No. of Spots.	P.M.	Recurrent Groups. No. of Groups.	P.M.	Revival Groups. No. of Groups.	P.M.
I.	1879–89	94	+°·14±·08	116	+°·05±·09		°
II.	1890–1900	145	+·06±·09	145	−·20±·09	212	+·16±·08
III.	1901–12	112	+·05±·08	119	−·03±·10	155	+·07±·07
IV.	1913–23	137	−·10±·07	124	−·36±·09	280	+·07±·06

* See also *M.N.R.A.S.*, **73**, 686.

The Recurrent groups were also divided up to test any relationship of proper motions with phase of the cycle, but a negative result was obtained. Finally, the Recurrent groups were classified into three approximate types: (1) First appearance, a stream; second appearance, one principal spot, sometimes with very small companions. (2) In both first and second appearances a stream of spots. (3) In both appearances single spots or small clusters of spots. As explained in the earlier part of the paper, types (1) might possibly give rise to a systematic drift equatorwards. No outstanding proper motion was shown in this classification, which points, at least, to a rough elimination of a source of error due to spot development.

Faculæ.—The accurate measurement of solar faculæ offers such great difficulties that one would hesitate to state the measures of proper motions unless the results were very consistent and point to appreciable drift in latitude. About 200 areas of faculæ had already been carefully selected from the Greenwich record for a determination of the Sun's rotation period.* The selection was based on the ability to recognise in succeeding rotations an area of faculæ of definite appearance, small or of moderate extent, unbroken activity, and sufficient isolation from other groups. The material so selected is included in the years 1888, 1898 to 1903, 1910 to 1916, 1919 to 1923—generally speaking, in years of small or moderate activity. The heliographic latitudes as well as the heliographic longitudes of these selected areas having been computed, the mean latitude in the first rotation, derived from some 4 or 6 measures, was compared with the mean latitudes in the following one. The intervals are therefore, roughly, equal to one synodic rotation, or about twenty-seven days. Any existing error of the position of the Sun's axis being very small, it can safely be neglected in the quantities comprising the following table; it was impossible, in fact, to arrange the measures otherwise, on account of their limited number, faculæ being measurable in most cases only near the Sun's limbs.

TABLE V.

Mean Proper Motions of Faculæ in an Interval of One Synodic Rotation.

	Latitudes.	Number of Groups.	Mean Random P.M.	Mean Systematic P.M.
	° °		°	°
Northern Hemisphere.	0 – 10	31	±1·0	+0·6
	10 – 20	50	±1·5	+0·7
	20 – 30	31	±1·9	+1·6
	30 – 40	5	±1·9	+1·9
Southern Hemisphere.	0 – 10	31	±1·0	−0·1
	10 – 20	24	±1·8	+1·2
	20 – 30	22	±1·8	+1·5
	30 – 40	7	±1·8	+1·3
N. and S. Hemispheres combined.	0 – 10	62	±1·0	+0·22 ± ·11
	10 – 20	74	±1·6	+0·85 ± ·14
	20 – 30	53	±1·8	+1·55 ± ·14
	30 – 40	12	±1·9	+1·6 ± ·2
All zones combined .	.	201	±1·49	+0·88 ± ·08

* *M.N.R.A.S.*, **84**, 431.

A number of these areas of faculæ were followed to the third rotation, this giving an interval of about fifty-five days for comparison of mean latitudes. These give measures independent of those tabulated above.

TABLE VI.

Mean Proper Motions in One Synodic Rotation derived from Long-Lived Faculæ.

	Latitudes.	Number of Groups.	Mean Random P.M.	Mean Systematic P.M.
	° °		°	°
N. and S. Hemispheres combined.	0 – 10	14	±0·6	+0·4
	10 – 20	18	±1·1	+0·9
	20 – 30	11	±1·6	+1·6
	30 – 40	Nil		
All zones combined . .		43	±1·0	+0·9 ±·1

Table V. supplemented by Table VI. appears to establish a marked poleward drift of faculæ amounting on the average, between latitudes 0° and 40°, to about 0°·9 per synodic rotation, or 0°·8 per sidereal rotation. The results for both hemispheres are consistent. This is satisfactory, and opposes the possibility that the drift is due to a systematic error in measurement, as the direction is reversed North and South of the Equator. Table V., in particular, also indicates an increasing systematic drift from the equatorial regions, where the drift is barely apparent, to zone 20° – 30°, where it amounts to 1½° per rotation. The long-lived faculæ support the idea that the drift is maintained. It is considered that the mean systematic proper motion represents a real drift of the groups of faculæ. These are generally more stable than spot groups, and are seen to disperse, as a whole, without a decided preference for preceding or following portion,* which might well give rise to an apparent drift.

Conclusions.

(1) The average random motion in latitude of single spots is very nearly 1° per sidereal rotation; for the centres of groups it is 1¼° both for the Recurrent and Revival groups.

(2) As regards a systematic latitude drift, single spots, which offer perhaps the most reliable material, do not support the existence of any such drift of as much as one-tenth of a degree per rotation. Neither is any drift established if the figures are combined for single spots, Recurrent groups, and Revivals—data which can reasonably be considered as supplementary. The Recurrent groups for latitudes 0° – 20° suggest an average drift towards the Equator of rather over two-tenths of a degree per rotation. The figures, however, are not conclusive, and the excess of *minus* signs is regarded as accidental, which would disappear with a longer series of observations. This conclusion is strengthened

* *M.N.R.A.S.*, **84**, 99.

by the results from Revival groups which are opposed in their trend of *plus* signs.

(3) It is interesting to note that the general progression equatorwards of the spot-zones during the course of a solar cycle (which on the average would amount to 0°·14 per sidereal rotation) cannot be traced in the motions of the individual spots.

(4) The consideration of the faculæ appears to establish the remarkable result that they are moving continuously away from the Equator, and that this movement is more pronounced in high than in low latitudes, reaching nearly $1\frac{1}{2}°$ per sidereal rotation in latitudes 20° – 30°.

Royal Observatory, Greenwich:
1924 *December* 11.

Occultation of B.D. – 18°6078 *by Mars observed on* 1924 *September* 5 *with the* 243-*mm. Merz-Repsold Refractor at the Astronomical Observatory, Kiew.* By Professor S. D. Tscherny.

(*Communicated by the Secretaries.*)

I. *Results of Micrometer Measures.*

Before the disappearance and after the reappearance of B.D. – 18°6078 I measured on September 5, with the filar micrometer (eyepiece with a magnifying power of 346), the position-angle (p) and the distance (s) of this star from the centre of Mars. The results of my micrometrical measures are as follows :—

	h	m	s		
(a)	22	0	44·79	K.S.T.	$s = 1^r·4535 = 41''·192$
	22	2	9·54	,,	$p = 263° 14'·62$
(b)	22	22	9·54	,,	$s = 1^r·001 = 28''·368$
	22	19	22·54	,,	$p = 262° 32'·50$
(c)	23	45	27·54	,,	$s = 0^r·670 = 18''·988$
	23	45	7·29	,,	$p = 94° 30'·75$

where the value of one revolution of micrometer screw at the temperature when the observations were made (+ 12°·5 R.) is 28″·3399.

Reducing the distances to the times of the observations of the position-angles, we have :

(a) $22^h\ 2^m\ 9^s·54$ K.S.T. $= 9^h\ 1^m\ 24^s·81$ G.M.T.	$s = 40''·353$	$p = 263° 14'·62$
Differential refraction	+0·012	−0·75
	40″·365	263° 13′·87

(*b*) $22^h\ 19^m\ 22^s{\cdot}54$ K.S.T. $= 9^h\ 18^m\ 34^s{\cdot}99$ G.M.T.	$s = 30''{\cdot}035$	$p = 262°\ 32'{\cdot}50$
Differential refraction	$+0{\cdot}007$	$-0{\cdot}84$
	$30''{\cdot}042$	$262°\ 31'{\cdot}66$
(*c*) $23^h\ 45^m\ 7^s{\cdot}29$ K.S.T. $= 10^h\ 44^m\ 5^s{\cdot}69$ G.M.T.	$s = 18''{\cdot}765$	$p = 94°\ 30'{\cdot}75$
Differential refraction	$+0{\cdot}010$	$+0{\cdot}62$
	$18''{\cdot}775$	$94°\ 31'{\cdot}37$

M. P. Ditschenko observed the star B.D. $-18°6078$ at my request with the 122-mm. Merz-Repsold Meridian Circle of our Observatory. The results of his observations are as follows:—

	α(1924·0).	δ(1924·0).
	h m s	° ′ ″
1924 Sept. 15	22 5 56·42	−18 22 3·00
18	56·46	1·43
20	56·68	2·58
21	56·46	3·17
22	56·35	2·75
23	56·25	1·92

The arithmetic mean of these observations is

$$\alpha = 22^h\ 5^m\ 56^s{\cdot}44\ ;\ \delta = -18°\ 22'\ 02''{\cdot}47.$$

Reducing these co-ordinates to the time of the micrometrical measures, we have

$$\alpha_{\text{app.}} = 22^h\ 5^m\ 59^s{\cdot}17\ ;\ \delta_{\text{app.}} = -18°\ 21'\ 44''{\cdot}53.$$

Based upon these co-ordinates of B.D. $-18°6078$ and upon my micrometrical measures, we have

	$\alpha_* - \alpha_\delta$.	$\delta_* - \delta_\delta$.
h m s	s	″
9 1 24·81 G.M.T.	−2·816	−4·758
9 18 34·99 ,,	−2·092	−3·907
10 44 5·69 ,,	+1·315	−1·480

Parallax of Mars (local *minus* geocentric place):

	α.	δ.
h m s	s	″
9 1 24·81 G.M.T.	+0·019	−21·11
9 18 34·99 ,,	−0·059	−21·11
10 44 5·69 ,,	−0·427	−20·69

Therefore the geocentric co-ordinates of Mars were

	α.	δ.
h m s	h m s	° ′ ″
9 1 24·81 G.M.T.	22 6 1·97	– 18 21 18·7
9 18 34·99 ,,	22 6 1·32	– 18 21 19·5
10 44 5·69 ,,	22 5 58·28	– 18 21 22·4

For the same times we have (*a*) from *C. d. T.* :

	α.	δ.
h m s	h m s	° ′ ″
9 1 24·81 G.M.T.	22 6 2·37	– 18 21 14·0
9 18 34·99 ,,	22 6 1·75	– 18 21 14·8
10 44 5·69 ,,	22 5 58·69	– 18 21 18·4

The corrections to Le Verrier's Tables of Mars are therefore

$$\Delta\alpha = -0^s{\cdot}40\,;\ -0^s{\cdot}43\,;\ -0^s{\cdot}41\,; \qquad \Delta\delta = -4''{\cdot}7\,;\ -4''{\cdot}7\,;\ -4''{\cdot}0,$$

or

$$\Delta\alpha = -0^s{\cdot}413. \qquad \Delta\delta = -4''{\cdot}47.$$

The corrections to Newcomb's Tables with Ross's corrections, as given in the *Nautical Almanac,* are

$$\Delta\alpha = +0^s{\cdot}03\,;\ +0^s{\cdot}00\,;\ +0^s{\cdot}02\,; \qquad \Delta\delta = -2''{\cdot}1\,;\ -2''{\cdot}1\,;\ -1''{\cdot}4,$$

or

$$\Delta\alpha = +0^s{\cdot}017\,; \qquad \Delta\delta = -1''{\cdot}87.$$

II. *Occultation of B.D.* – 18°6078 *by Mars.*

During the occultation Mars was near the horizon ($h = 19^\circ{\cdot}5$). The eyepiece of magnifying power 346 was used. As the star approached to the bright disc of Mars its brightness decreased.

The results of my observations are as follows :—

h m s	
9 47 44·20 G.M.T.	Star very faint.
9 48 59·99 ,,	Disappearance. Limb of Mars boiling.
10 35 39·33 ,,	Reappearance. Good. Star very faint. Limb of Mars boiling.

Between the disappearance and the reappearance I measured with the filar micrometer the polar diameter of Mars. The results of my micrometrical measures are as follows :—

$10^h\ 4^m$ G.M.T. $\quad t = +12^\circ{\cdot}5$ R.

$0^r{\cdot}928\,;\ 0^r{\cdot}912\,;\ 0^r{\cdot}921\,;\ 0^r{\cdot}902\,;\ 0^r{\cdot}919\,;\ 0^r{\cdot}904\,;\ 0^r{\cdot}914\,;$
$0^r{\cdot}921\,;\ 0^r{\cdot}913\,;\ 0^r{\cdot}919.$

The arithmetic mean is $0^r{\cdot}9153$.

Therefore the diameter of Mars for $10^h\ 4^m$ G.M.T. is $25''{\cdot}940$, with

the probable error $\pm 0''\cdot048$. The difference between this value of diameter of Mars and his value calculated from the *N.A.* is $+1''\cdot76$.

From the times of disappearance and reappearance, the co-ordinates of Mars (*N.A.*), observed semidiameter of Mars $= 12''\cdot97$ and hor. par. $= 22''\cdot72$ (*N.A.*), I deduced the following equations of condition :—

$$\text{D.} \quad [3\cdot66997]_n\Delta\alpha + [3\cdot28000]_n\Delta\delta = [3\cdot11992]''$$
$$\text{R.} \quad [3\cdot73338]\ \Delta\alpha + [3\cdot09827]_n\Delta\delta = [3\cdot95660]''.$$

These equations give :

$\Delta\alpha = +0''\cdot964 = +0^s\cdot064$; $\Delta\delta = -3''\cdot057$ for Newcomb's Tables, with Ross's corrections, and

$\Delta\alpha = -0^s\cdot366$; $\Delta\delta = -5''\cdot66$ for the Tables of Le Verrier (*C. d. T.*).

Astronomical Observatory, Kiew :
1924 *November* 21.